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THE REDUCTION OF STAR SETS

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Mahler's theory of irreducible star bodies is redeveloped and extended in a modified form. It is shown that any closed bounded star set S contains a closed irreducible star set T having the same critical determinant. Further, it is shown that, if the first set S is bounded by a finite number of algebraic surfaces, then there will be an irreducible set T which is also bounded by a finite number of algebraic surfaces.

1. Introduction

If $A_1 = (a_1^{(1)}, \ldots, a_n^{(1)}), \ldots, A_n = (a_1^{(n)}, \ldots, a_n^{(n)})$ are any linearly independent points in n-dimensional space, the set of all points of the form

$$u_1 A_1 + \ldots + u_n A_n = (u_1 a_1^{(1)} + \ldots + u_n a_1^{(n)}, \ldots, u_1 a_n^{(1)} + \ldots + u_n a_n^{(n)}),$$

where $u_1, \ldots, u_n$ are arbitrary integers, is called a lattice, the lattice generated by $A_1, \ldots, A_n$. It is well known that, if Λ is any lattice generated both by the points $A_1, \ldots, A_n$ and by the points $B_1, \ldots, B_n$, then the corresponding determinants $|a_r^{(s)}|$, $|b_r^{(s)}|$ have the same absolute value; this common absolute value of the determinants is called the determinant of the lattice and is denoted by $d(\Lambda)$.

Mahler (1946*a*, 1949) says that a lattice Λ is 'admissible' for a set S, if there is no point of Λ, with the possible exception of the origin O, in the interior of S. Mahler defines the critical determinant of a set S to be the lower bound of the determinants of the lattices 'admissible' for S, the critical determinant having the value $+\infty$ if there are no lattices admissible for S. Whereas Mahler uses $\Delta(S)$ to denote the critical determinant defined in this way, we shall denote it by $\Delta_M(S)$.

The problem of determining $\Delta_M(S)$ for various sets S is the central problem of the geometry of numbers. This problem was discussed in detail by Minkowski (1904) in the case when S is a two- or three-dimensional convex region symmetrical in O. General discussions of the problem have been given by Mahler (1946*a*, *b*) in the case when S is a star body.† More recently Mahler (1949) has discussed the problem for general sets S.

Mahler makes a special study of the critical lattices of a set. He defines a critical lattice of a set S to be a lattice Λ 'admissible' for S with $d(\Lambda) = \Delta_M(S)$. In particular, he proves (Mahler 1949, theorem 2) that, if S contains O as an inner point, and if $\Delta_M(S)$ is finite, then S has at least one critical lattice.

† For the definition of a star body see Mahler (1946*a*). It is not difficult to prove that Mahler's definition is equivalent to the statement that a star body is a closed set S, such that, if X is a point of S, then all points of the form λX with $|\lambda| < 1$ are inner points of S.

Mahler says that a star body S is reducible if there is a star body T with $\Delta_M(T) = \Delta_M(S)$ which is a proper subset of S; otherwise S is irreducible. It is clear that the irreducible star bodies are of particular interest. Mahler (1946*b*, 1947) proves that they have various special properties and makes a detailed study of their critical lattices (see also Rogers 1947*a*).

A natural and indeed a fundamental problem in the theory is stated by Mahler (1946*b*, p. 450) as his

Problem 7. *To decide whether every bounded reducible star body contains at least one irreducible star body of equal determinant.*

Mahler makes the following remark on this problem:

'It is highly probable that the answer is in the affirmative, and that even a continuous infinity of irreducible star bodies of the wanted kind exists; but I have not succeeded in proving this. One reason for this failure is the following fact: If H, K, K_1, K_2, ... are star bodies such that

$$\Delta(K) = \Delta(K_1) = \Delta(K_2) = \dots,$$

and

$$K \supset K_1 \supset K_2 \supset \dots \supset H,$$

then the star bodies K_r tend to a limiting set, namely, their intersection, but this set is not necessarily a star body. Presumably, a proof will be constructive and will consist of a finite number of steps.... If Problem 7 has an affirmative answer, then only irreducible star bodies need be considered for most purposes, in so far as bounded star bodies are concerned.... The analogous problem for unbounded star bodies has probably a negative answer; but again, I have not so far succeeded in proving this.'

We decide this problem in appendix 2 by giving an example of a bounded reducible star body which contains no irreducible star body of equal determinant. The example shows clearly that the difficulty mentioned by Mahler is insuperable as long as we confine our attention to star bodies. The main object of this paper is to show that, if we admit a slightly larger class of sets and use slightly modified definitions, we can develop a theory, very similar to that of Mahler, but in which we can prove the desired result. The proof is of the type suggested by Mahler; it is constructive and consists of a finite number of steps.

We now explain the way in which we modify Mahler's theory. A lattice Λ will be said to be (strictly) admissible† for a set S, if there is no point of Λ, with the possible exception of O, in S. The lower bound of the determinants $d(\Lambda)$ of the lattices Λ (strictly) admissible for a set S will be called the (modified) critical determinant of S and will be denoted by $\Delta(S)$, the value of $\Delta(S)$ being $+\infty$ if there are no such lattices. Comparing these definitions with those of Mahler, it is plain that $\Delta(S) = \Delta_M(S)$ if S is open; further, this equality also holds when S is a star body, since then

$$\Delta_M(S) \leqslant \Delta(S) < \Delta_M((1+\epsilon)\,S) = (1+\epsilon)^n\,\Delta_M(S)$$

for any positive ϵ. However, in appendix 1 we give some examples of sets of points for which $\Delta_M(S) < \Delta(S)$. There is certainly a very close connexion between the functionals Δ and Δ_M. It is clear that $\Delta_M(S) = \Delta(S_0)$, where S_0 is the set of inner points of S. On the other hand, it is easy to see that

$$\Delta(S) = \inf_T \Delta_M(T) = \inf_T \Delta(T),$$

† The term '(strictly) admissible' is used in the introduction to avoid confusion with Mahler's terminology; in the later sections the parenthesized adverb will be omitted.

the lower bound being taken over all open sets T containing S; since, if Λ is a lattice which is (strictly) admissible for S, then the set T of all points of the space, which are not points other than O of Λ, is an open set containing S with

$$\Delta(S) \leqslant \Delta_M(T) = \Delta(T) = d(\Lambda).$$

A lattice Λ will be called a critical lattice of S if $d(\Lambda) = \Delta(S)$ and Λ is the limit lattice of a convergent sequence† of lattices $\Lambda^{(1)}, \Lambda^{(2)}, \ldots$, all (strictly) admissible for S. Note that if $d(\Lambda) = \Delta(S)$ and Λ is (strictly) admissible for S, then Λ is, according to this definition, a critical lattice of S, since then all the lattices $\Lambda^{(1)}, \Lambda^{(2)}, \ldots$ may be taken to coincide with Λ. It is clear that this definition is equivalent to that of Mahler when S is open and also when S is a star body. Mahler has proved a number of important theorems concerning the critical lattices of a set. In § 3 of this paper we use Mahler's methods to show that some of these results have close analogues when our definitions are used in place of his.

We use the word 'reducible' in a slightly wider sense than that used by Mahler. A set S will be said to be reducible to a set T, if T is a proper subset of S and $\Delta(T) = \Delta(S)$. A set S will be said to be reducible among a class of sets when S can be reduced to a set T in the class; and, conversely, S will be said to be irreducible among the class of sets when there is no set T of the class to which S may be reduced. In this terminology it is clear that Mahler confines his attention to the process of reduction among the star bodies. We shall be concerned in this paper with the processes of reduction among the *star sets* and among the *proper star sets*. By a star set we shall understand a set S of points, which is closed, and which has the property that if X is any point of S then the point λX also belongs to S for $-1 \leqslant \lambda \leqslant 1$. We shall call a star set S proper, if it is the closure of the set of its inner points, and if it contains O as an inner point. In § 4 we obtain some results analogous to the results of Mahler for reduction among the star bodies. In addition, we prove the following result, having no analogue in Mahler's theory.

THEOREM 7. *Let S be any bounded star set with $\Delta(S) > 0$. If S is reducible among the star sets, then S can be reduced to a proper star set T which is irreducible among the star sets.*

While this theorem provides a fairly complete solution of the problem of Mahler quoted above, there is a sense in which the theorem has the disadvantage of being too general. There is little or no point in starting with a set S bounded by an algebraic surface and reducing it to a set which may be bounded by a highly pathological surface. In § 5 we give the phrase 'a closed set bounded by a finite number of algebraic surfaces' a precise meaning, and call such a set an algebraic complex. We prove the following theorem:

THEOREM 12. *Let S be a proper bounded star set, which is also an algebraic complex, and which is reducible among the star sets. Then S can be reduced to a proper star set T, which is an algebraic complex, and which is irreducible among the star sets.*

Examples of non-convex irreducible two-dimensional star domains‡ have been given by Mahler (1944, 1946*b*, § 7), Ollerenshaw (1945*a*, § 7) and Cassels (1947). All these examples are of star domains bounded by a finite number of algebraic curves. I know of no example of a non-convex irreducible three-dimensional star body. But by applying theorem 12 to known results we have the following existence theorem:

† See Mahler (1946*a*) or § 2 below.
‡ A star domain is a two-dimensional star body.

THEOREM 13. *The star bodies $S_1, \ldots, S_6$, defined by the inequalities shown in table 1 have the critical determinants stated. For $r = 1, \ldots, 6$ the body S_r contains a proper bounded star set T_r, which is an algebraic complex, which is irreducible among the star sets, and which is such that*

$$\Delta(T_r) = \Delta(S_r).$$

TABLE 1

body	defining inequalities	critical determinant	dimension of space
S_1	$\lvert x_1 \rvert + \lvert x_2 \rvert + \lvert x_3 \rvert \leqslant 1$	$\frac{19}{108}$	3
S_2	$\lvert x_1 x_2 x_3 \rvert \leqslant 1$	7	3
S_3	$\lvert x_1(x_2^2+x_3^2) \rvert \leqslant 1$	$\sqrt{\frac{23}{4}}$	3
S_4	$\lvert x_1^2+x_2^2-x_3^2 \rvert \leqslant 1$	$\sqrt{\frac{3}{2}}$	3
S_5	$\lvert x_1^2+x_2^2+x_3^2-x_4^2 \rvert \leqslant 1$	$\sqrt{\frac{7}{4}}$	4
S_6	$\lvert x_1^2+x_2^2-x_3^2-x_4^2 \rvert \leqslant 1$	$\frac{3}{2}$	4

An explicit construction for any of the star sets $T_1, \ldots, T_6$, whose existence is established in theorem 13, would be of considerable interest. Work of Davenport (1941) suggests that T_2 may possibly be the region defined by

$$\left.\begin{array}{c} \lvert x_1 x_2 x_3 \rvert \leqslant 1 \\ (x_2-x_3)^2+(x_3-x_1)^2+(x_1-x_2)^2 \leqslant 14; \end{array}\right\}$$

while work of Mullineux (1951) suggests that T_4 may possibly be the region defined by

$$\lvert x_1^2+x_2^2-x_3^2 \rvert \leqslant 1,$$
$$\lvert x_3 \rvert \leqslant \sqrt{2}.$$

When one considers convex star bodies, the results provided by theorems 7 and 12 leave much to be desired. One is loath to replace a convex body by a non-convex set and is interested in reducing a bounded convex star body, if possible, to a star body which is both convex and irreducible among the star bodies. In the two-dimensional case Mahler (1947) has given a characterization of all convex irreducible star domains and has proved the fundamental theorem that *every two-dimensional convex star domain K contains an irreducible convex star domain H (not necessarily different from K) with $\Delta(H) = \Delta(K)$*. I hope to prove in another paper by a refinement of Mahler's methods that, *if the original convex star domain K is bounded by a finite number of algebraic arcs, then the irreducible convex star domain H can always be chosen so that it is also bounded by a finite number of algebraic arcs*. But the problem of generalizing Mahler's result to more dimensions seems to be difficult. It is easy to prove by Mahler's methods that *every convex star body K contains a convex star body H, which is irreducible among the convex star bodies*; but I do not see how to prove (as is necessary) that every convex star body H, which is irreducible among the convex star bodies, is also irreducible among the star bodies.

2. LATTICES

We first summarize certain definitions and results of Mahler (1946*a*). A sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is said to be *bounded*, if the sequence of determinants $d(\Lambda_1), d(\Lambda_2), \ldots$ is bounded, and there is a number $\epsilon > 0$ such that for $r = 1, 2, \ldots$ there is no point X other than O of Λ_r satisfying $\lvert X \rvert < \epsilon$ (Mahler 1946*a*, definition 1). Here $\lvert X \rvert$ denotes, as usual, the distance of X from O. A sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is said to *converge*† to a limit lattice Λ, if it is possible to find points $A_1, \ldots, A_n$ generating Λ and points $A_1^{(r)}, \ldots, A_n^{(r)}$ generating Λ_r, for $r = 1, 2, \ldots$, such that

$$A_i^{(r)} \to A_i \quad (i = 1, \ldots, n)$$

as $r \to \infty$. It follows immediately from this definition that, *if the sequence of lattices* $\Lambda_1, \Lambda_2, \ldots$ *converges to the lattice* Λ, *then* $d(\Lambda_r) \to d(\Lambda)$ *as* $r \to \infty$. Mahler (1946*a*, theorem 2) proves the fundamental theorem that *every bounded sequence of lattices contains a convergent subsequence.*

The following lemma is implicit in Mahler's work (1946*a*, the proof of theorem 19):

Lemma 1. *Let* $X_1, \ldots, X_n$ *be points generating a lattice* Λ, *and satisfying* $|X_r| < M$ *for* $r = 1, \ldots, n$. *If* $u_1, \ldots, u_n$ *are any integers and*

$$X = u_1 X_1 + \ldots + u_n X_n, \tag{1}$$

then

$$|u_r| \leqslant \frac{M^{n-1}|X|}{d(\Lambda)} \quad \textit{for } r = 1, \ldots, n. \tag{2}$$

Proof. The determinant of the lattice generated by the points

$$X_1, \ldots, X_{r-1}, X, X_{r+1}, \ldots, X_n$$

is $|u_r|\, d(\Lambda)$. But by Hadamard's inequality this determinant is less than or equal to

$$|X_1| \ldots |X_{r-1}| . |X| . |X_{r+1}| \ldots |X_n|.$$

Hence

$$|u_r|\, d(\Lambda) \leqslant M^{n-1} |X|,$$

so that (2) is satisfied.

Our next lemma is a simple consequence of lemma 1; again it is implicit in Mahler's work (1946*a*, the proof of theorem 19).

Lemma 2. *Let* $X_1, \ldots, X_n$ *be linearly independent points; let* R *and* ϵ *be positive numbers. Then there exists a positive number* η *with the following property. For all points* $Y_1, \ldots, Y_n$ *with*

$$|X_r - Y_r| < \eta \quad (r = 1, \ldots, n), \tag{3}$$

and for all integers $u_1, \ldots, u_n$, *if the points*

$$\left.\begin{aligned} X &= u_1 X_1 + \ldots + u_n X_n, \\ Y &= u_1 Y_1 + \ldots + u_n Y_n, \end{aligned}\right\} \tag{4}$$

satisfy either $|X| \leqslant R$ *or* $|Y| \leqslant R$, *then* $|X - Y| < \epsilon$.

Proof. Let Λ be the lattice generated by the points $X_1, \ldots, X_n$. Let M be so large that

$$|X_r| < \tfrac{1}{2}M, \quad \text{for } r = 1, \ldots, n.$$

Choose η satisfying

$$0 < \eta < \tfrac{1}{2}M, \tag{5}$$

$$\eta < \frac{d(\Lambda)}{2n(n!)\, M^{n-1}} \tag{6}$$

and

$$\eta < \frac{\epsilon d(\Lambda)}{2nRM^{n-1}}. \tag{7}$$

Let $Y_1, \ldots, Y_n$ be any points satisfying (3) and let Λ' be the lattice generated by these points. Then

$$|Y_r| < M, \quad \text{for } r = 1, \ldots, n,$$

and it is easy to verify, by use of (6), that

$$d(\Lambda') > d(\Lambda) - \tfrac{1}{2}d(\Lambda) = \tfrac{1}{2}d(\Lambda).$$

† Although this definition of convergence is not identical with that of Mahler (1946*a*, definition 2), it is easily seen to be equivalent to his definition.

Now suppose that $u_1, \dots, u_n$ are any integers. First suppose that the point Y given by (4) satisfies $|Y| \leqslant R$. Then by lemma 1

$$|u_r| \leqslant \frac{RM^{n-1}}{d(\Lambda')} < \frac{2RM^{n-1}}{d(\Lambda)},$$

for $r = 1, \dots, n$. Consequently

$$|X - Y| \leqslant \sum_{r=1}^{n} |u_r| \,.\, |X_r - Y_r| < \frac{2nRM^{n-1}}{d(\Lambda)} \eta < \epsilon,$$

on using (7). If $|X| \leqslant R$, the same argument (with the roles of X and Y interchanged) shows that $|X - Y| < \epsilon$. This proves the lemma.

The next lemma provides an alternative definition for the convergence of a sequence of lattices; it is deduced from lemma 2 using methods due to Mahler (see Mahler 1946*a*, the proof of theorems 18 and 19).

LEMMA 3. *The sequence of lattices $\Lambda_1, \Lambda_2, \dots$ converges to the limit lattice Λ, if, and only if, both*

(*a*) *each point of Λ is the limit point of a convergent sequence of points $X_1, X_2, \dots$ belonging respectively to $\Lambda_1, \Lambda_2, \dots$, and*

(*b*) *each limit point of each sequence of points $X_1, X_2, \dots$ belonging respectively to $\Lambda_1, \Lambda_2, \dots$ is a point of Λ.*

Proof. Suppose that the lattices $\Lambda_1, \Lambda_2, \dots$ converge to the lattice Λ. Then it is possible to find points $A_1, \dots, A_n$ generating Λ and points $A_1^{(r)}, \dots, A_n^{(r)}$ generating Λ_r, for $r = 1, 2, \dots$, such that

$$A_i^{(r)} \to A_i \quad (i = 1, \dots, n)$$

as $r \to \infty$. So, if $X = u_1 A_1 + \dots + u_n A_n$ is any point of Λ, the point

$$X_r = u_1 A_1^{(r)} + \dots + u_n A_n^{(r)}$$

of Λ_r converges to X as r tends to infinity. On the other hand, if the point

$$X_r = u_1^{(r)} A_1^{(r)} + \dots + u_n^{(r)} A_n^{(r)}$$

of Λ_r converges to a point X as r tends to infinity through a strictly increasing sequence of positive integers, then by lemma 2 it is clear that the corresponding point

$$Y_r = u_1^{(r)} A_1 + \dots + u_n^{(r)} A_n$$

of Λ also converges to X as r tends to infinity through the sequence of positive integers, so that X is in Λ as Λ is closed. Thus the conditions (*a*) and (*b*) are both satisfied.

Suppose that the condition (*a*) is satisfied. Then we can choose points $A_1, \dots, A_n$ generating Λ and points $A_1^{(r)}, \dots, A_n^{(r)}$ of Λ_r, for $r = 1, 2, \dots$, such that

$$A_i^{(r)} \to A_i \quad (i = 1, \dots, n)$$

as $r \to \infty$. Let us suppose that the sequence of lattices $\Lambda_1, \Lambda_2, \dots$ does not converge to the lattice Λ. Then it is clear that we can choose a strictly increasing sequence $r_1, r_2, \dots$ of positive integers such that for $r = r_1, r_2, \dots$ the points $A_1^{(r)}, \dots, A_n^{(r)}$ are not a basis for Λ_r. It is clear that when r is sufficiently large the points $A_1^{(r)}, \dots, A_n^{(r)}$ will be linearly independent. So we may suppose that $A_1^{(r)}, \dots, A_n^{(r)}$ are linearly independent for $r = r_1, r_2, \dots$. Then for these values of r we can find a point

$$X_r = \mu_1^{(r)} A_1^{(r)} + \dots + \mu_n^{(r)} A_n^{(r)}$$

of Λ_r, for which $\mu_1^{(r)}, \ldots, \mu_n^{(r)}$ are not all integers. By replacing X_r by a point of the form

$$X_r - u_1^{(r)} A_1^{(r)} - \ldots - u_n^{(r)} A_n^{(r)},$$

where $u_1^{(r)}, \ldots, u_n^{(r)}$ are integers, we may suppose that

$$0 < \max\{|\mu_1^{(r)}|, \ldots, |\mu_n^{(r)}|\} \leqslant \tfrac{1}{2}.$$

By replacing the new point X_r by a point of the form $2^t X_r$, where t is an integer, we may suppose that

$$\tfrac{1}{4} < \max\{|\mu_1^{(r)}|, \ldots, |\mu_n^{(r)}|\} \leqslant \tfrac{1}{2}.$$

By replacing the sequence $r_1, r_2, \ldots$ by a subsequence of itself, we may suppose that

$$\mu_i^{(r)} \to \mu_i \quad (i = 1, \ldots, n)$$

as $r \to \infty$ through the subsequence. Then

$$\tfrac{1}{4} \leqslant \max\{|\mu_1|, \ldots, |\mu_n|\} \leqslant \tfrac{1}{2}, \tag{8}$$

and the point X_r of Λ_r converges to the point

$$X = \mu_1 A_1 + \ldots + \mu_n A_n$$

as $r \to \infty$ through the subsequence. But it is clear from (8) that X is not a point of Λ. Thus the condition (b) is not satisfied. Hence if the conditions (a) and (b) are both satisfied it follows that the sequence $\Lambda_1, \Lambda_2, \ldots$ converges to Λ. This completes the proof of the lemma.

3. Critical lattices

In this section we use the definitions of the introduction and by application of Mahler's methods we prove three theorems about critical lattices, which are analogous to the results Mahler has proved using his definitions. If S is any set we use $\tilde{S}$ to denote the set formed by adding the origin O to the union

$$\bigcup_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m} S. \tag{9}$$

Our first result is a refinement of results of Mahler (1946a, theorem 8, and 1949, theorem 2).

Theorem 1. *Let S be any set with $\Delta(S) < +\infty$. If O is an inner point of $\tilde{S}$, then S has at least one critical lattice.*

Proof. As $\Delta(S) < +\infty$, there is at least one lattice admissible for S. It is clear from the definition of $\Delta(S)$ that we can choose a sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ (not necessarily distinct) such that Λ_r is admissible for S ($r = 1, 2, \ldots$), and such that $d(\Lambda_r) \to \Delta(S)$ as $r \to \infty$.

On the supposition that O is an inner point of $\tilde{S}$, we can choose $\rho > 0$ so small that the sphere

$$|X| < \rho \tag{10}$$

is contained in $\tilde{S}$. If X were any point, other than O, of Λ_r in the sphere (10), then X would be in the set (9), and so for some integer $m \neq 0$ the point mX would be in S. As Λ_r is admissible for S, it follows that there is no point of Λ_r, except O, in the sphere (10). Thus the sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is bounded. So by a fundamental result of Mahler, stated in § 2 above, the lattice Λ_r will converge to some lattice Λ^* as r tends to infinity through a suitable sequence $r_1, r_2, \ldots$ of positive integers. Since $d(\Lambda_r) \to d(\Lambda^*)$ as r tends to infinity through this subsequence, we see that $d(\Lambda^*) = \Delta(S)$. Thus Λ^* is a critical lattice of S and the theorem is proved.

Before we prove our next result it is convenient to obtain the following elementary lemma:

LEMMA 4. *If $R>0$, $\Delta>0$ and A is a point with*

$$0<|A|<\frac{\Delta}{R^{n-1}},$$

there is a lattice Λ with $d(\Lambda)<\Delta$, having A as a lattice point, and such that the only points of Λ in the sphere $|X|<R$ are of the form mA, where $m=0, \pm 1, \pm 2, \ldots$.

Proof. By the spherical symmetry we may suppose without loss of generality that A is the point with co-ordinates $(a, 0, \ldots, 0)$, where $0<a<\Delta/R^{n-1}$. Then it is clear that the lattice Λ generated by the points

$$\begin{gathered}(a,\ 0,\ \ldots,\ 0),\\(0,\ R,\ \ldots,\ 0),\\\ldots\\(0,\ 0,\ \ldots,\ R),\end{gathered}$$

has the required properties.

The following result is a refinement of another result of Mahler (1949, theorem 3):

THEOREM 2. *Let S be a bounded set. Then $\Delta(S)$ is finite. Further*

(*a*) *S has at least one critical lattice if and only if $\Delta(S)>0$, and*

(*b*) *$\Delta(S)>0$ if and only if O is an inner point of $\tilde{S}$.*

Proof. As S is bounded it is trivial that there are lattices admissible for S, so that $\Delta(S)$ is finite. In order to prove (*a*) and (*b*) it clearly suffices, in view of theorem 1, to prove the following two auxiliary results.

(i) If S has a critical lattice, then $\Delta(S)>0$.

(ii) If $\Delta(S)>0$, then O is an inner point of $\tilde{S}$.

To prove (i) we note that, if Λ is a critical lattice of S, then $\Delta(S)=d(\Lambda)>0$.

To prove (ii) suppose that $\Delta(S)>0$, while O is not an inner point of $\tilde{S}$. Choose R so large that each point X of S satisfies $|X|<R$. Choose a point A with

$$0<|A|<\frac{\Delta(S)}{R^{n-1}},$$

which is not in $\tilde{S}$. Then none of the points of the form

$$mA \quad (m=\pm 1, \pm 2, \ldots),$$

are in S. So it follows by lemma 4 that there is a lattice Λ with $d(\Lambda)<\Delta(S)$ with no point, except perhaps O, in S. This contradiction to the definition of $\Delta(S)$ proves (ii).

The last result of this section is a generalization of a result which is well known for bounded star bodies (see Mahler 1946*a*, theorem 11).

THEOREM 3. *If Λ is a critical lattice of a bounded set S, then there are n linearly independent points of Λ on the boundary of S.*

Proof. Let α be the smallest linear manifold containing O and the points (if any) of Λ on the boundary of S. We suppose that the dimension k of α is less than n and obtain a contradiction. Choose points $X_1, \ldots, X_n$ generating Λ so that the points $X_1, \ldots, X_k$ (if $k\geq 1$) generate all the points of Λ lying in α.

As Λ is a critical lattice of S, it is the limit of a sequence of lattices $\Lambda_1, \Lambda_2, \ldots$, admissible for S. So by lemma 3 for $r=1, 2, \ldots$ we can choose points $X_1^{(r)}, \ldots, X_n^{(r)}$ of Λ_r such that

$$X_i^{(r)} \to X_i \quad \text{as } r\to\infty$$

for $i = 1, \ldots, n$. Since $X_1, \ldots, X_n$ generate Λ and

$$d(\Lambda_r) \to d(\Lambda) \quad \text{as } r \to \infty,$$

it is clear that $X_1^{(r)}, \ldots, X_n^{(r)}$ generate Λ_r, if r is sufficiently large.

Choose R so large that all the points X of S satisfy $|X| \leqslant R$. Now there are only a finite number of points, $A_1, \ldots, A_l$, say, of Λ, which are in the sphere $|X| \leqslant R+1$, but which are not in α. These points are not on the boundary of S. So we can choose ϵ, with $0 < \epsilon < 1$, so small that for every point X of S we have

$$|X - A_r| > \epsilon, \quad \text{for } r = 1, \ldots, l. \tag{11}$$

By lemma 2 we can choose a number $\eta > 0$ so small that, for all points $Z_1, \ldots, Z_n$ with

$$|X_i - Z_i| < 2\eta \quad (i = 1, \ldots, n),$$

and for all integers $u_1, \ldots, u_n$, if the points

$$X = u_1 X_1 + \ldots + u_n X_n,$$
$$Z = u_1 Z_1 + \ldots + u_n Z_n,$$

satisfy $|Z| \leqslant R$, then $|X - Z| < \epsilon$.

Choose M so large that

$$|X_i| < M, \quad \text{for } i = 1, \ldots, n.$$

Now by taking r sufficiently large and by writing

$$Y_1 = X_1^{(r)}, \ldots, Y_n = X_n^{(r)},$$

we can find points $Y_1, \ldots, Y_n$, generating a lattice Λ' admissible for S, such that

$$|X_i - Y_i| < \eta, \quad \text{for } i = 1, \ldots, n,$$

and

$$d(\Lambda') < d(\Lambda)(1+\delta)^{n-k},$$

where

$$\delta = \frac{\eta}{\eta + M}.$$

Write

$$Z_1 = Y_1, \ldots, Z_k = Y_k, \ Z_{k+1} = (1-\delta) Y_{k+1}, \ldots, Z_n = (1-\delta) Y_n.$$

Then, for $i = 1, \ldots, n$,

$$|Z_i - X_i| \leqslant |Y_i - X_i| + \delta |Y_i| < \eta + \delta(M + \eta) = 2\eta.$$

Further, if Λ^* is the lattice generated by the points $Z_1, \ldots, Z_n$, we have

$$d(\Lambda^*) = (1-\delta)^{n-k} d(\Lambda') < (1-\delta^2)^{n-k} d(\Lambda) < d(\Lambda) = \Delta(S). \tag{12}$$

Now consider any lattice point

$$Z = u_1 Z_1 + \ldots + u_n Z_n$$

of Λ^* and the corresponding points

$$X = u_1 X_1 + \ldots + u_n X_n,$$
$$Y = u_1 Y_1 + \ldots + u_n Y_n$$

of Λ and Λ'. If $|Z| > R$, the point Z is certainly not in S. If $|Z| \leqslant R$, we have $|Z - X| < \epsilon$ by our choice of η. If $u_{k+1}, \ldots, u_n$ are not all zero, X is not in α and satisfies $|X| < R + \epsilon < R + 1$, and so X is one of the points $A_1, \ldots, A_l$. In this case it follows by (11) that Z is not in S. If

$u_{k+1} = \ldots = u_n = 0$, then Z coincides with the point Y of Λ', and so, as Λ' is admissible for S, the point Z can only be in S if it coincides with O. This shows that Λ^* is admissible for S. Now (12) is contrary to the definition of $\Delta(S)$. This contradiction proves the theorem.

4. Irreducibility among the star sets and among the proper star sets

We recall that a star set S is a set which is closed, and which has the property that if X is any point of S then the point λX also belongs to S for $-1 \leqslant \lambda \leqslant 1$. Further, a proper star set S is a star set which contains O as an inner point, and which is the closure of its set of inner points. Again a star body S is a closed set, which has the property that, if X is a point of S, then all points of the form λX with $-1 < \lambda < 1$ are inner points of S. Clearly a star body is a proper star set and a proper star set is a star set. Example 3 of appendix 1 is an example of a proper star set which is not a star body, while example 2 of that appendix is an example of a star set which is not a proper star set.

In this section we consider the processes of reduction among the star sets and among the proper star sets. We need three definitions.

Definition 1. *A point X of a set S is said to be a primitively irreducible point if X is a point of a lattice Λ with $d(\Lambda) < \Delta(S)$ such that the only points of Λ in S are of the form*

$$mX, \quad \text{where } m = 0, \pm 1, \pm 2, \ldots.$$

Definition 2.† *A point X of a set S is said to be an irreducible point if X is primitively irreducible or if X is a limit point of primitively irreducible points.*

Definition 3. *A point X of a set S is said to be an outer boundary point of S if, for all numbers $\lambda > 1$, the point λX is not in S.*

Although these definitions have been stated for an arbitrary set S, we shall confine our attention in the sequel to star sets; indeed it is probable that these definitions are only appropriate in this case.

We need two lemmas. The first is a simple consequence of our definitions.

Lemma 5. *Let S be any set and let T be any star set, which is contained in S, and which has the same critical determinant as S. Then T contains every irreducible point of S and every such point is an irreducible point of T.*

Proof. Let X_0 be any primitively irreducible point of S. Then there is a lattice Λ with

$$d(\Lambda) < \Delta(S) = \Delta(T)$$

such that the only points of Λ in S are of the form

$$mX_0, \quad \text{where } m = 0, \pm 1, \pm 2, \ldots.$$

Since T is a star set contained in S with $\Delta(T) > d(\Lambda)$ it follows that X_0 must be in T. So we see that X_0 is a primitively irreducible point of T. As T is closed it follows that every irreducible point of S is an irreducible point of T.

The second lemma is proved by a method I have used to prove a rather similar lemma (Rogers 1947*a*).

Lemma 6. *Let T be a star set properly contained in a proper star set S with $\Delta(S) < +\infty$. Then there is an outer boundary point of S which is not in T.*

† This definition of an irreducible point differs from another definition I have used (Rogers 1947*a*); the definitions are equivalent in the case when S is a star body and X is a boundary point of S.

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Proof. We suppose that every outer boundary point of S belongs to T and we obtain a contradiction. As T is properly contained in S there is a point X_1 of S which is not in T. As T is closed there is a whole sphere with centre X_1 which contains no point of T. As S is the closure of its interior, there is an inner point X_2 of S in the interior of this sphere with centre X_1. So, if $\epsilon > 0$ is sufficiently small, the sphere

$$|X - X_2| \leqslant \epsilon \tag{13}$$

is contained in S but contains no point of T.

Let X be any point in the sphere (13); it is in S but not in T. If for some positive λ the point λX is not in S, then for some μ with $1 < \mu < \lambda$ the point μX is an outer boundary point of S which is not in T. This is contrary to our original supposition. Hence for every positive λ the point λX is in S. Thus for all positive λ the set S contains the sphere of points $Y = \lambda X$ satisfying

$$|Y - \lambda X_2| \leqslant \lambda \epsilon. \tag{14}$$

But if Λ is any lattice, λ can be chosen so large that the sphere given by (14) contains a point other than O of Λ. Consequently there is no lattice which is admissible for S, and so $\Delta(S) = +\infty$. This contradiction of our hypotheses proves the lemma.

We use these two lemmas to prove the following theorem, analogous to one of my previous theorems (1947*a*, theorem 1), giving a necessary and sufficient condition for a proper star set to be irreducible among the star sets or among the proper star sets:

THEOREM 4. *Let S be a proper star set with $\Delta(S) < +\infty$. Then*

(*a*) *S is irreducible among the proper star sets if and only if every outer boundary point of S is irreducible; and*

(*b*) *S is irreducible among the star sets if and only if every outer boundary point of S is irreducible.*

Proof. It clearly suffices to prove the following three auxiliary results:

(i) If every outer boundary point of S is irreducible, then S is irreducible among the star sets.

(ii) If S is irreducible among the star sets, then S is irreducible among the proper star sets.

(iii) If S is irreducible among the proper star sets, every outer boundary point of S is irreducible.

To prove (i) we suppose that every outer boundary point of S is irreducible but that S can be reduced to a star set T, and we obtain a contradiction. It follows from lemma 6 that there is an outer boundary point X_0, say, of S which is not in T. Now X_0 is an irreducible point of S, and so it follows by lemma 5 that $\Delta(T) < \Delta(S)$, contrary to our supposition that S could be reduced to T. This proves (i).

The result (ii) follows immediately from the definitions.

To prove (iii) we suppose that S is irreducible among the proper star sets, and we consider an arbitrary outer boundary point X_0 of S. As S contains O as an inner point, we can choose a number δ, satisfying $0 < \delta < 1$, so small that S contains the set of all points X with

$$|X| < \delta |X_0|.$$

Since the point $(1 + \frac{1}{4}\delta) X_0$ is not in the closed set S, we can choose a number η, with $0 < \eta < \frac{1}{4}\delta$, so small that there is no point X of S satisfying both

$$|X| \geqslant (1 + \tfrac{1}{4}\delta) |X_0|$$

and $$\left|\frac{X}{|X|}-\frac{X_0}{|X_0|}\right|\leqslant\eta.$$

Let T_0 be the set of all inner points X of S, which satisfy either

$$|X|<(1-\tfrac{1}{4}\delta)\,|X_0|$$

or $$\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|>\eta$$

for both signs. Let T be the closure of the open set T_0. Then T is a proper star set properly contained in S. Thus, as S is irreducible among the proper star sets, we have $\Delta(T)<\Delta(S)$. Consequently there is a lattice Λ admissible for T with $d(\Lambda)<\Delta(S)$. Since $d(\Lambda)<\Delta(S)$ there is a point X_1 other than O of Δ in S. As Λ is admissible for T, the point X_1 is not in the closure T of T_0. But, since S is a proper star set, X_1 is a limit point of inner points of S. This X_1 is a limit point of inner points of S which do not belong to T_0. Hence X_1 is a limit point of points X satisfying both $$|X|\geqslant(1-\tfrac{1}{4}\delta)\,|X_0|$$

and $$\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|\leqslant\eta$$

for some sign. It follows that there is a point $X_2=\pm X_1$ of Λ in S satisfying both

$$|X_2|\geqslant(1-\tfrac{1}{4}\delta)\,|X_0|$$

and $$\left|\frac{X_2}{|X_2|}-\frac{X_0}{|X_0|}\right|\leqslant\eta.$$

By our choice of η, we must have

$$(1-\tfrac{1}{4}\delta)\,|X_0|\leqslant|X_2|<(1+\tfrac{1}{4}\delta)\,|X_0|.$$

Now, using the last two inequalities,

$$\begin{aligned}|X_2-X_0|&=|X_0|\left|\frac{X_2}{|X_2|}-\frac{X_0}{|X_0|}+\frac{|X_2|-|X_0|}{|X_2|\,.\,|X_0|}X_2\right|\\&\leqslant|X_0|\left|\frac{X_2}{|X_2|}-\frac{X_0}{|X_0|}\right|+\big||X_2|-|X_0|\big|\\&\leqslant(\eta+\tfrac{1}{4}\delta)\,|X_0|<\tfrac{1}{2}\delta\,|X_0|.\end{aligned}\tag{15}$$

Similarly, if X_3 were any point of Λ in S other than O and $\pm X_2$, there would be a point $X_4=\pm X_3$ of Λ other than O and $\pm X_2$ in S satisfying

$$|X_4-X_0|<\tfrac{1}{2}\delta\,|X_0|.$$

But then the point $X_5=X_4-X_2$ would be a point other than O of Λ, satisfying

$$|X_5|=|X_4-X_2|<\delta\,|X_0|,$$

and so would be a point other than O of Λ in T. This is impossible as Λ is admissible for T. Hence the only points of Λ in S are the points $-X_2$, O and X_2, while $d(\Lambda)<\Delta(S)$. Thus the point X_2 is a primitively irreducible point of S satisfying (15). Since δ and η may be taken to be arbitrarily small, it follows that X_0, being a limit point of primitively irreducible points X_2 of S, is an irreducible point of S. This proves (iii) and completes the proof of the theorem.

Corollary 1. *Let S be a star body with $\Delta(S) < +\infty$. Then S is irreducible among the star bodies if and only if it is irreducible among the star sets or the proper star sets.*

Proof. The result follows by the theorem and theorem 1 of Rogers (1947*a*) using the equivalence for star bodies of the present definitions with those adopted there.

Corollary 2. *Let S be a star set with $\Delta(S) < +\infty$, containing O as an inner point. If S is irreducible among the star sets then every outer boundary point of S is irreducible.*

Proof. This corollary can easily be proved by the method used to prove the auxiliary result (iii) in the proof of theorem 4. The modifications that are necessary are obvious simplifications; it is no longer necessary to ensure that T is a proper star set.

We next want to prove a result analogous to one of Mahler (1946*b*, theorem C) on the critical lattices of an irreducible star body. But first we prove the following lemma:

Lemma 7. *Let S be a star set with $\Delta(S) < +\infty$, containing O as an inner point. Let X_0 be any irreducible outer boundary point of S. Then there is a critical lattice Λ of S having X_0 as a lattice point.*

Proof. Since X_0 is an outer boundary point of S it is clear that, for each positive integer l, the point

$$\frac{l+1}{l} X_0$$

is not a point of the closed set S, so that for a suitable $\epsilon = \epsilon_l$ with $0 < \epsilon < 1/l$ there will be no point X of S satisfying

$$|X| \geq \frac{l+1}{l} |X_0|$$

and

$$\left| \frac{X}{|X|} - \frac{X_0}{|X_0|} \right| < \epsilon.$$

Since X_0 is an irreducible point of S, there is a primitively irreducible point X_l of S satisfying

$$\big| |X_l| - |X_0| \big| < \frac{1}{l} |X_0|$$

and

$$\left| \frac{X_l}{|X_l|} - \frac{X_0}{|X_0|} \right| < \epsilon.$$

But X_l, being a primitively irreducible point of S, is a lattice point of a lattice Λ_l, with $d(\Lambda_l) < \Delta(S)$, such that the only points of Λ in S are of the form

$$mX_l, \quad \text{where } m = 0, \pm 1, \pm 2, \ldots.$$

For each integer $l > 1$ we consider the lattice

$$\Lambda'_l = \frac{l+1}{l-1} \Lambda_l.$$

If X is a point other than O of Λ'_l which is in S, it is clear that

$$X = m \frac{l+1}{l-1} X_l$$

for some non-zero integer m. Further

$$|X| \geq \frac{l+1}{l-1} |X_l| \geq \frac{l+1}{l-1} \left(1 - \frac{1}{l}\right) |X_0| = \frac{l+1}{l} |X_0|$$

and

$$\left| \frac{X}{|X|} - \frac{X_0}{|X_0|} \right| = \left| \frac{X_l}{|X_l|} - \frac{X_0}{|X_0|} \right| < \epsilon;$$

and so, by our choice of ϵ, the point X cannot be in S. Consequently Λ'_l is admissible for S. Thus

$$\Delta(S) \leqslant d(\Lambda'_l) = \left(\frac{l+1}{l-1}\right)^n d(\Lambda_l) < \left(\frac{l+1}{l-1}\right)^n \Delta(S)$$

and

$$\lim_{l\to\infty} d(\Lambda'_l) = \Delta(S).$$

As O is an inner point of S and Λ'_l is admissible for S, the sequence of lattices $\Lambda'_2, \Lambda'_3, \ldots$ is bounded. So, by Mahler's convergence theorem, this sequence contains a subsequence converging to some limit lattice Λ. Since

$$d(\Lambda) = \lim_{l\to\infty} d(\Lambda'_l) = \Delta(S),$$

while the lattices $\Lambda'_2, \Lambda'_3, \ldots$ are admissible for S, it follows that Λ is a critical lattice of S. But it is clear that, as l tends to infinity, the point

$$\frac{l+1}{l-1} X_l$$

of Λ'_l converges to the point X_0. So by lemma 3 we see that X_0 is a point of the critica. lattice Λ. This proves the lemma.

The following analogue of Mahler's result (1946*b*, theorem C) is an immediate consequence of the above lemma and corollary 2 to theorem 4.

THEOREM 5. *Let S be a star set with $\Delta(S) < +\infty$, containing O as an inner point. If S is irreducible among the star sets, then every outer boundary point of S is a lattice point of some corresponding critical lattice of S.*

For the rest of this section we confine our attention to bounded star sets. We first prove the following result concerning the set of primitively irreducible points of such a set.

THEOREM 6. *Let S be any bounded star set with $\Delta(S) > 0$. Then the set of primitively irreducible points of S is open and contains all points other than O of some sphere with centre O.*

Proof. Choose R so large that

$$|X| < R$$

for every point X of S. Let X_1 be any primitively irreducible point of S. Choose Λ to be a lattice with $d(\Lambda) < \Delta(S)$, having X_1 as a lattice point, such that the only points of Λ in S are of the form

$$mX_1, \quad \text{where } m = 0, \pm 1, \pm 2, \ldots. \tag{16}$$

We may suppose that Λ is generated by points $X_1, X_2, \ldots, X_n$. Now there are only a finite number of points Y of Λ satisfying

$$|Y| < R+1.$$

Let $Y_1, \ldots, Y_h$ be the set of all such points of Λ which are not of the form (16). Then, as none of these points are in the closed set S, we can choose a number ϵ, with $0 < \epsilon < 1$, such that

$$|X - Y_l| > \epsilon, \quad \text{for } l = 1, \ldots, h,$$

for every point X of S. By lemma 2, all sufficiently small positive numbers η have the following property. For every point X_1^* with

$$|X_1^* - X_1| < \eta, \tag{17}$$

and for all integers $u_1, \ldots, u_n$, if

$$Y = u_1 X_1 + u_2 X_2 + \ldots + u_n X_n, \tag{18}$$

$$Y^* = u_1 X_1^* + u_2 X_2 + \ldots + u_n X_n, \tag{19}$$

and $$|Y^*| < R, \quad \text{then} \quad |Y^* - Y| < \epsilon.$$

We now show that every point X_1^* satisfying (17) is a primitively irreducible point of S, provided η is sufficiently small. Consider any such point X_1^*. Provided η is sufficiently small the points $X_1^*, X_2, \ldots, X_n$ are linearly independent and generate a lattice Λ^* with

$$d(\Lambda^*) < \Delta(S).$$

Consider any point Y^* of Λ^* which is not of the form

$$mX_1^*, \quad \text{where } m = 0, \pm 1, \pm 2, \ldots. \tag{20}$$

Then Y^* is of the form (19) where $u_1, \ldots, u_n$ are integers, $u_2, \ldots, u_n$ not being all zero. If $|Y^*| \geqslant R$, then Y^* is not in S. Suppose that $|Y^*| < R$ and let Y be the corresponding point of the form (18). Then, provided η is sufficiently small, we conclude that $|Y^* - Y| < \epsilon$. In particular, it follows that

$$|Y| < |Y^*| + \epsilon < R + 1.$$

As $u_2, \ldots, u_n$ are not all zero, Y is not of the form (16) and so $Y = Y_l$ for some integer l with $1 \leqslant l \leqslant h$. Thus $|Y^* - Y_l| < \epsilon$, and it follows from our choice of ϵ that Y^* is not a point of S. This shows that the only points of Λ^* in S are of the form (20). Since $d(\Lambda^*) < \Delta(S)$ it follows that X_1^* is a point of S and that X_1^* is indeed a primitively irreducible point of S. This completes the proof that the set of primitively irreducible points of X is open.

It follows immediately from lemma 1 that every point X with

$$0 < |X| < \frac{\Delta(S)}{R^{n-1}} \tag{21}$$

is a primitively irreducible point of S. This completes the proof of the theorem.

Corollary. *Let S be a bounded star set, with $\Delta(S) > 0$, which is irreducible among the star sets. Then S is a proper star set.*

Proof. By theorem 6 the set of primitively irreducible points of S is open and contains all points other than O of some sphere with centre O. So S contains O as an inner point, and, by corollary 2 to theorem 4, every outer boundary point of S is irreducible. Thus every outer boundary point of S being irreducible is a limit point of inner points of S. Since S is a bounded star set it follows that every point of S is a limit point of inner points of S, so that S is a proper star set.

Our next object is to prove the main theorem of this section, namely, theorem 7 (stated in § 1), but we first need two lemmas.

Lemma 8. *Let S be a bounded star set. Suppose that $|X| < R$ for all points X of S. Let ϵ be a real number with $0 < \epsilon < \Delta(S)/R^{n-1}$. Suppose that for some point X_0 with $0 < |X_0| \leqslant R$, all outer boundary points X of S with*

$$\left.\begin{aligned} &|X| > |X_0|, \\ &\left|\frac{X}{|X|} \pm \frac{X_0}{|X_0|}\right| < \frac{\epsilon}{4R} \quad (\textit{for some sign}), \end{aligned}\right\} \tag{22}$$

are irreducible. Then there is a star set T contained in S with $\Delta(T)=\Delta(S)$ *and such that all the outer boundary points X of T with*

$$\left.\begin{aligned} &|X|>|X_0|-\tfrac{1}{4}\epsilon,\\ &\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}\quad(\textit{for some sign}),\end{aligned}\right\}\tag{23}$$

are irreducible.

Proof. We take T to be the union of the set T_1 of all points X of S such that either

$$|X|\leqslant|X_0|-\tfrac{1}{4}\epsilon$$

or

$$\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|\geqslant\frac{\epsilon}{4R}\quad(\text{for both signs}),$$

and the set T_2 of all points of the form λX, where $0\leqslant\lambda\leqslant1$ and X is an irreducible point of S. Then clearly T is a star set contained in S.

We prove that $\Delta(T)=\Delta(S)$. As S contains T, we have $\Delta(T)\leqslant\Delta(S)$. We suppose that $\Delta(T)<\Delta(S)$ and eventually obtain a contradiction. As $\Delta(T)<\Delta(S)$, there is a lattice Λ with $d(\Lambda)<\Delta(S)$ and with no point other than O in T. Since $d(\Lambda)<\Delta(S)$, there is a point X_1 other than O of Λ in S. Since X_1 is not in T, we must have

$$\left.\begin{aligned} &|X_1|>|X_0|-\tfrac{1}{4}\epsilon,\\ &\left|\frac{X_1}{|X_1|}\pm\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}\quad(\text{for some sign}).\end{aligned}\right\}$$

Replacing X_1 by $-X_1$, if necessary, we may suppose that

$$\left|\frac{X_1}{|X_1|}-\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}.$$

Since X_1 is in S but is not in T, there is a number $\lambda\geqslant1$ such that $Y=\lambda X_1$ is an outer boundary point of S which is not an irreducible point of S. As Y satisfies

$$\left|\frac{Y}{|Y|}-\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R},$$

it is clear from our hypotheses that $|Y|\leqslant|X_0|$. Hence $|X_1|\leqslant|X_0|$.

Suppose that X_2 is any point other than $-X_1$, O and X_1 of Λ in S. Then X_2 is not in T, and replacing X_2 by $-X_2$, if necessary, we may suppose that

$$|X_2|>|X_0|-\tfrac{1}{4}\epsilon,$$

$$\left|\frac{X_2}{|X_2|}-\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}.$$

As before we must have $|X_2|\leqslant|X_0|$. Now

$$\begin{aligned}|X_2-X_1|&=|X_0|\left|\left(\frac{X_2}{|X_2|}-\frac{X_0}{|X_0|}\right)-\left(\frac{X_1}{|X_1|}-\frac{X_0}{|X_0|}\right)+\left(\frac{|X_2|-|X_0|}{|X_2|.|X_0|}\right)X_2-\left(\frac{|X_1|-|X_0|}{|X_1|.|X_0|}\right)X_1\right|\\ &\leqslant R\left\{\left|\frac{X_2}{|X_2|}-\frac{X_0}{|X_0|}\right|+\left|\frac{X_1}{|X_1|}-\frac{X_0}{|X_0|}\right|\right\}+||X_2|-|X_0||+||X_1|-|X_0||\\ &\leqslant\epsilon<\frac{\Delta(S)}{R^{n-1}}.\end{aligned}$$

So by lemma 4 the point X_2-X_1 is a primitively irreducible point of S and so belongs to T. But this is impossible since X_2-X_1 is a point other than O of a lattice Λ admissible for T. Hence the only points of Λ in S are $-X_1$, O and X_1. Consequently X_1 is a primitively irreducible point of S and so is a point of T. This second contradiction proves that $\Delta(T)=\Delta(S)$.

Now let X be any outer boundary point of T satisfying (23). It is clear from the construction of T that $X=\lambda Y$, where $0\leqslant\lambda\leqslant 1$ and Y is an irreducible point of S. But, since $\Delta(T)=\Delta(S)$ and T is contained in S, it follows by lemma 5 that Y is an irreducible point of T. As X is an outer boundary point of T, we must have $\lambda=1$, and so $X=Y$ is an irreducible point of T. This completes the proof of the lemma.

LEMMA 9. *Let S be a bounded star set. Suppose that $|X|<R$ for all points X of S. Let ϵ be a real number with $0<\epsilon<\Delta(S)/R^{n-1}$. Let X_0 be any point other than O. Then there is a star set T contained in S, with $\Delta(T)=\Delta(S)$ and such that all the outer boundary points X of T with*

$$\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}\quad(\textit{for some sign})\tag{24}$$

are irreducible.

Proof. The lemma is unaffected if we replace X_0 by λX_0 for any $\lambda\neq 0$. So we suppose that for some positive integer N,

$$N|X_0|=R,$$

and that

$$|X_0|<\tfrac{1}{4}\epsilon.$$

Write

$$X_h=hX_0$$

for $h=1,2,\ldots,N$. We use lemma 8 to give an inductive construction of a sequence of sets $T_N, T_{N-1},\ldots,T_0$. We take $T_N=S$. Then as there are no outer boundary points X of T_N with $|X|>|X_N|=R$, there is by lemma 8 a star set T_{N-1} contained in T_N with $\Delta(T_{N-1})=\Delta(T_N)$ and such that every outer boundary point X of T_{N-1} satisfying

$$\left.\begin{aligned}&|X|>|X_{N-1}|>|X_N|-\tfrac{1}{4}\epsilon,\\&\left|\frac{X}{|X|}\pm\frac{X_N}{|X_N|}\right|<\frac{\epsilon}{4R}\quad(\text{for some sign}),\end{aligned}\right\}$$

is irreducible. Proceeding inductively in this way we can construct star sets $T_N, T_{N-1},\ldots,T_0$ such that

$$T_0\subset T_1\subset\ldots\subset T_N,$$

$$\Delta(T_0)=\Delta(T_1)=\ldots=\Delta(T_N)=\Delta(S),$$

and that every outer boundary point X of T_h satisfying

$$\left.\begin{aligned}&|X|>h|X_0|,\\&\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}\quad(\text{for some sign}),\end{aligned}\right\}$$

is irreducible, for $h=0,\ldots,N$. Now it is clear that T_0 has the required properties for the set T.

Proof of theorem 7.† Choose R so large that every point X of S satisfies $|X|<R$. Choose a number ϵ with $0<\epsilon<\Delta(S)/R^{n-1}$. Choose points $X_1, X_2,\ldots,X_N$ such that

$$|X_1|=|X_2|=\ldots=|X_N|=1,$$

† See p. 61.

and such that every point X other than O satisfies

$$\left|\frac{X}{|X|} \pm \frac{X_r}{|X_r|}\right| < \frac{\epsilon}{4R} \quad \text{(for some sign)}, \tag{25}$$

for some integer r with $1 \leqslant r \leqslant N$. By applying lemma 9 repeatedly we can choose star sets $T_1, \ldots, T_N$ with

$$\left.\begin{aligned} S \supset T_1 \supset T_2 \supset \ldots \supset T_N, \\ \Delta(S) = \Delta(T_1) = \Delta(T_2) = \ldots = \Delta(T_N), \end{aligned}\right\} \tag{26}$$

and such that, for $r = 1, \ldots, N$, all the outer boundary points X of T_r satisfying (25) are irreducible.

Take $T = T_N$ and consider any outer boundary point X of T. Let r be the integer with $1 \leqslant r \leqslant N$ such that X satisfies (25). As X is in T it is in T_r and so for some $\lambda \geqslant 1$ the point $Y = \lambda X$ is an outer boundary point of T_r. It follows from our construction of T_r that Y is an irreducible point of T_r. Hence, by lemma 5, Y is an irreducible point of T, and so Y coincides with X. Thus every outer boundary point of T is irreducible.

Now, if T_0 is any star set contained in T with $\Delta(T_0) = \Delta(T)$, then, by lemma 5, we see that every outer boundary point of T belongs to T_0. Since T is bounded this implies that T and T_0 coincide. Thus T is irreducible among the star sets. Consequently, by our hypotheses, T does not coincide with S, and S can be reduced to the star set T which is irreducible among the star sets. Further, by the corollary to theorem 6, the set T is necessarily a proper star set. This completes the proof of the theorem.

As a direct consequence of this theorem we have the following result showing that the critical determinant of any bounded star set can be determined from a knowledge of the critical determinant of a corresponding proper star set.

THEOREM 8. *Let S be any bounded star set. Then S has the same critical determinant as the proper star set T defined to be the closure of the set of inner points of S.*

Proof. If $\Delta(S) = 0$ the result is trivial. If $\Delta(S) > 0$, then, by theorem 7, S can be reduced to a proper star set S', with $\Delta(S') = \Delta(S)$. But clearly

$$S' \subset T \subset S,$$

so that

$$\Delta(S') \leqslant \Delta(T) \leqslant \Delta(S).$$

Thus $\Delta(T) = \Delta(S)$ and the theorem is proved.

We remark that the second example in appendix 1 shows that the condition that S is bounded is needed in theorem 8.

5. REDUCTION OF BOUNDED ALGEBRAIC STAR SETS

Before we can prove the main theorem of this section, namely, theorem 12 stated in § 1, we have to introduce some definitions and to prove some results on the combinatorial topology of regions bounded by algebraic surfaces. The various concepts will be explained in detail and elementary proofs will be given for the results.

We first introduce the concepts† of algebraic gratings, cells and complexes. If $F(x_1, \ldots, x_n)$ is a polynomial in $x_1, \ldots, x_n$ with real coefficients, which is not identically zero, the set of all points X in n-dimensional space with co-ordinates $(x_1, \ldots, x_n)$ satisfying

$$F(x_1, \ldots, x_n) = 0$$

is called an algebraic grating or for short a grating. Clearly a grating is a closed set.

† Our terminology is based on that used by Newman (1939).

Let G be an algebraic grating and let X_0 be any point not in G. Let S_0 be the set of all points X which can be connected to X_0 by a continuous curve which does not meet G; and let S be the closure of S_0. Any such set S is by definition called an algebraic n-cell of G or more simply a cell of G; the corresponding set S_0 is called the domain of the cell. It is clear that a cell S of a grating G is a closed connected set, and that the corresponding set S_0 is the set of points of S which are not in G and is an open connected set. The union of any number of cells of an algebraic grating G is called an algebraic n-complex on G or a complex on G.

THEOREM 9†. *Let G be any algebraic grating in the n-dimensional space* $\mathscr{R}$ *of points X with coordinates* $(x_1, \ldots, x_n)$. *Then the number of cells of G is finite. Further, if* $n \geqslant 2$, *there is an algebraic grating G′ in the* $(n-1)$*-dimensional space* $\mathscr{R}'$ *of points X′ with co-ordinates* $(x_1, \ldots, x_{n-1}, 0)$ *such that the closure of the orthogonal projection of any cell of G on the space* $\mathscr{R}'$ *is a complex in* $\mathscr{R}'$ *on G′.*

Proof. We prove the result by an induction on n. When $n = 1$, the grating G reduces to a set of a finite number (possibly 0) of points on a line. The cells of G are the closed connected sets on this line, with points of G as end points, containing no points of G in their interior; it is clear that G has only a finite number of cells.

We now suppose that $n \geqslant 2$ and that every algebraic grating in $(n-1)$-dimensional space has only a finite number of cells. Let G be any algebraic grating in the n-dimensional space $\mathscr{R}$. We suppose that G is the grating defined by the polynomial $F(x_1, \ldots, x_n)$; then $F(x_1, \ldots, x_n)$ is not identically zero. We write $F(x_1, \ldots, x_n) = F(X)$ for convenience.

We need some well-known algebraic results. A polynomial in $x_1, \ldots, x_n$ with real coefficients is said to be irreducible if it cannot be expressed as the product of two non-constant polynomials with real coefficients. Now by a fundamental theorem of algebra, the polynomial $F(x_1, \ldots, x_n)$ can be decomposed into its real factors in essentially one way.‡ Let $H(X)$ be the polynomial formed from $F(X)$ by multiplying together those irreducible real factors of $F(X)$ which are essentially distinct (two factors being regarded as essentially the same if one is a constant multiple of the other). Then $H(X)$ has no non-constant factor which is a perfect square. Further $H(X) = 0$ if and only if $F(X) = 0$, and the grating defined by the polynomial $H(X)$ is the grating G defined by $F(X)$.

We may write

$$H(X) = \sum_{r=0}^{m} h_r(X')\, x_n^r \tag{27}$$

for some non-negative integer m, where $h_0(X'), \ldots, h_m(X')$ are polynomials in $x_1, \ldots, x_{n-1}$ with real coefficients, the polynomial $h_m(X')$ not being identically zero. Let $D(X')$ be the discriminant of the polynomial (27) regarded as a polynomial in x_n. Then $D(X')$ is a polynomial in the coefficients $h_0(X'), \ldots, h_m(X')$ and so is a polynomial in $x_1, \ldots, x_{n-1}$ with real coefficients. Further, by a well-known algebraic result, § if $D(X')$ were identically zero, then $H(X)$ would have a factor which was the square of a non-constant polynomial in

† [*Note added in proof* (20 *May* 1952). Theorem 9 and its consequence theorem 10 are not really original. I find that van der Waerden (1930) has proved a theorem very similar to theorem 9. But the theorems differ in a number of ways and it seems desirable to give a detailed proof of the result in the form in which it is required in the sequel.]

‡ See, for example, van der Waerden (1940, §23).

§ See van der Waerden (1940, §§27, 28); van der Waerden works with polynomials over a field, but it is easy to see that the result we require holds in any integral domain having unique factorization.

$x_1, \ldots, x_n$ with real coefficients. Thus it follows from our choice of $H(X)$ that $D(X')$ is not identically zero.

We write
$$J(X') = h_m(X')\,D(X')$$
and we take G' to be the grating defined by the polynomial $J(X')$ in the $(n-1)$-dimensional space $\mathscr{R}'$. This is permissible as $J(X')$ is not identically zero.

Let K be any cell of G and take C to be the sum of all the cells of G' whose domain contains some point of the orthogonal projection K' of K on the space $\mathscr{R}'$ (i.e. the set K' of all points $X' = (x_1, \ldots, x_{n-1})$ such that for some x_n the point $X = (x_1, \ldots, x_n)$ is in K). Clearly C is a complex on the algebraic grating G'. We eventually prove that C is in fact the closure of K'. We first prove that C contains the closure of K'. By our inductive hypothesis G' has only a finite number of cells. So C is the sum of a finite number of closed sets and is thus closed. Let A be any point of K, and let A' be the projection of A on $\mathscr{R}'$. Then A is the limit point of a sequence $X_1, X_2, \ldots$ of inner points of K, and the projections $X_1', X_2', \ldots$ of $X_1, X_2, \ldots$ are inner points of K' converging to A'. But there are points Y' of $\mathscr{R}'$ not on G' arbitrarily near to the point X_r', for $r = 1, 2, \ldots$. Since any point Y' of $\mathscr{R}'$ not on G' is in the domain of some cell of G', and all points Y' of $\mathscr{R}'$ sufficiently near to X_r' are in K', it follows that there are points Y' of C arbitrarily near to X_r'. Thus A' is a limit point of limit points $X_1, X_2, \ldots$ of C. As C is closed it follows that A' is in C. This proves that K' is contained in C, and so C contains the closure of K'.

Now consider any cell S of the grating G' contained in the complex C. We shall prove that every point in the domain S_0 of S is the projection of some point of K. This will prove that C is contained in the closure of K'. As S is a cell of C, there is a point X' of the domain S_0 of S, which is a point of K', and which is therefore the projection of a point X of K. We can clearly choose a point A of the domain K_0 of K so close to X that the projection A' of A is in the domain S_0. Let B' be any other point of the domain S_0. Our object† is to prove that B' is the projection of a point B of K_0.

Since A' and B' are in the domain S_0 of the cell S of G', there is a continuous curve in the space $\mathscr{R}'$, which does not meet G', joining A' to B'. This means that there are continuous real functions $x_1(t), \ldots, x_{n-1}(t)$, such that
$$X'(0) = (x_1(0), \ldots, x_{n-1}(0), 0) = A' = (a_1, \ldots, a_{n-1}, 0),$$
$$X'(1) = (x_1(1), \ldots, x_{n-1}(1), 0) = B' = (b_1, \ldots, b_{n-1}, 0),$$
and
$$J(X'(t)) = J(x_1(t), \ldots, x_{n-1}(t)) \neq 0$$
for $0 \leqslant t \leqslant 1$. For each t with $0 \leqslant t \leqslant 1$, let $k(t)$ be the number of real roots of the equation
$$H(X'(t), \xi) = H(x_1(t), \ldots, x_{n-1}(t), \xi) = \sum_{r=0}^{m} h_r(X'(t))\,\xi^r = 0, \tag{28}$$
regarded as an equation in ξ, and let $\xi_1(t), \ldots, \xi_{k(t)}(t)$ be the real roots of this equation arranged in order so that
$$\xi_1(t) < \xi_2(t) < \ldots < \xi_{k(t)}(t).$$
As $J(X'(t)) \neq 0$ for $0 \leqslant t \leqslant 1$, we have $h_m(X'(t)) \neq 0$ for $0 \leqslant t \leqslant 1$, and so $|h_m(X'(t))|$ has a positive lower bound for $0 \leqslant t \leqslant 1$. Again $|h_0(X'(t))|, \ldots, |h_m(X'(t))|$ have finite upper

† Our method is based on a method used by Ostrowski (1920).

bounds for $0 \leqslant t \leqslant 1$. Thus for some M all the real and complex roots ξ of (28) satisfy $|\xi| < M$ for $0 \leqslant t \leqslant 1$. Further as $D(X'(t)) \neq 0$ for $0 \leqslant t \leqslant 1$, the modulus

$$|D(X'(t))\{h_m(X'(t))\}^{2-2n}|$$

of the product of the squares of the differences between the roots of (28) has a positive lower bound for $0 \leqslant t \leqslant 1$, and so there is an $\epsilon > 0$ so small that

$$\xi_{i+1}(t) - \xi_i(t) > \epsilon, \quad \text{for } i = 1, \ldots, k(t)-1,$$

for $0 \leqslant t \leqslant 1$.

Consider any number t with $0 \leqslant t \leqslant 1$. Choose any δ with $0 < \delta < \epsilon$. Choose an integer h and numbers

$$\eta_j = \eta_0 + j\delta \quad (j = 0, 1, \ldots, h),$$

so that

$$\eta_0 < -M < M < \eta_h$$

and

$$H(X'(t), \eta_j) \neq 0, \quad \text{for } j = 0, 1, \ldots, h.$$

Choose $\tau > 0$ so small that for $j = 0, 1, \ldots, h$ the numbers

$$H(X'(t), \eta_j), \quad H(X'(s), \eta_j)$$

have the same sign for all s with $0 \leqslant s \leqslant 1$ and $|s-t| < \tau$. Now, as $\delta < \epsilon$, for any such value of s (including the value $s = t$) the interval $\eta_{j-1} \leqslant \xi < \eta_j$ can contain neither a multiple real root nor a pair of real roots of the equation

$$H(X'(s), \xi) = 0, \tag{29}$$

and contains just one real root if and only if the signs of the numbers

$$H(X'(s), \eta_{j-1}), \quad H(X'(s), \eta_j)$$

are different, i.e. if and only if the signs of the numbers

$$H(X'(t), \eta_{j-1}), \quad H(X'(t), \eta_j)$$

are different. Further every real root of the equation (29) lies in one of the intervals

$$\eta_{j-1} \leqslant \xi < \eta_j \quad (j = 1, \ldots, h).$$

Consequently $k(s) = k(t)$ and

$$|\xi_i(t) - \xi_i(s)| < \delta, \quad \text{for } i = 1, \ldots, k(t),$$

provided $0 \leqslant s \leqslant 1$ and $|s-t| < \tau$. Hence $k(t)$ is a constant, say k, for $0 \leqslant t \leqslant 1$ and

$$\xi_1(t), \ldots, \xi_k(t)$$

are continuous functions of t for $0 \leqslant t \leqslant 1$.

Write

$$\xi_0(t) = \min\{a_n - 1, -M\},$$

$$\xi_{k+1}(t) = \max\{a_n + 1, M\},$$

for $0 \leqslant t \leqslant 1$. Since A is not a point of G we have

$$H(X'(0), a_n) = H(A) \neq 0,$$

and so we can choose an integer i with $0 \leqslant i \leqslant k$ such that

$$\xi_i(0) < a_n < \xi_{i+1}(0).$$

We write $$x_n(t) = \lambda\xi_i(t) + \mu\xi_{i+1}(t)$$
for $0 \leqslant t \leqslant 1$, where λ and μ are constants with $\lambda+\mu = 1$, $\lambda > 0$, $\mu > 0$ chosen so that
$$x_n(0) = a_n.$$
It is clear that $$H(X(t)) = H(X'(t), x_n(t)) \neq 0$$
for $0 \leqslant t \leqslant 1$. Take B to be the point with co-ordinates $(b_1, \ldots, b_{n-1}, b_n)$, where $b_n = x_n(1)$. Then the curve given parametrically by
$$X = X(t) = (x_1(t), \ldots, x_n(t)) \quad (0 \leqslant t \leqslant 1)$$
is a continuous curve leading from A to B and further $H(X) \neq 0$ for each point X of this curve. Thus A and B are joined by a continuous curve which does not meet G and so B is a point of the domain K_0. Hence B', the projection of B, is a point of K'. This proves that any point of the domain of any cell of the complex C is in K'. It follows that every point of C is in the closure of K'. This completes the proof that C is the closure of K'. Thus the closure of the orthogonal projection of any cell of G on the space $\mathscr{R}'$ is a complex in $\mathscr{R}'$ on G'.

Consider any cell S of the grating G'. Suppose that $K_1, \ldots, K_h$ are distinct cells of the grating G whose projections $K'_1, \ldots, K'_h$ have points $X'_1, \ldots, X'_h$ in the domain S_0 of S. Then we can choose points $A_1, \ldots, A_h$ in the domains of the cells $K_1, \ldots, K_h$ with projections $A'_1, \ldots, A'_h$ so close to $X'_1, \ldots, X'_h$ that $A'_1, \ldots, A'_h$ are in S_0. Then, as in the last paragraphs, for any point B' of S_0 there are points $B_1, \ldots, B_h$ in the domains of the cells $K_1, \ldots, K_h$ whose projection is B'. Now, if $1 \leqslant i < j \leqslant h$, the points B_i and B_j are in the domains of the distinct cells K_i, K_j of G, and so there is at least one point of G on the line segment joining B_i to B_j. Consequently there are at least $h-1$ points of G on the line through $B', B_1, \ldots, B_h$, and the equation
$$H(B', \xi) = \sum_{r=0}^{m} h_r(B')\,\xi^r = 0$$
has at least $h-1$ real roots for ξ. But as B' is not in G' we have $h_m(B') \neq 0$, and this equation has at most m real roots. Thus $h \leqslant m+1$. Hence there are at most $m+1$ cells of G whose projection has a point in the domain of any particular cell of G'. As the projection of each cell of G certainly has a point in the domain of some cell of G', and as G' has only a finite number of cells, it follows that the number of cells of G is less than or equal to $m+1$ times the number of cells of G'.

We have now completed the proof of the theorem in n-dimensional space on the assumption that the first assertion is true in $(n-1)$-dimensional space. The result follows by induction.

THEOREM 10. *Let K be a complex on an algebraic grating G. Let T be the closure of the set of all points of the form λX where $0 \leqslant \lambda \leqslant 1$ and X is in K. Then T is a complex on some algebraic grating G_1.*

Proof. Let G be the grating defined by a polynomial
$$F(X) = F(x_1, \ldots, x_n)$$
of degree m. Consider the grating G' in the $(n+1)$-dimensional space of points
$$X' = (x_1, \ldots, x_n, x_{n+1})$$
defined by the polynomial
$$H(X') = (x_{n+1})^{m+1}(1-x_{n+1})\,F\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right).$$

Let K' be the union of those cells of G' whose domain contains a point of the form $\frac{1}{2}A'$, where $A' = (a_1, \ldots, a_n, 1)$ and the point $A = (a_1, \ldots, a_n)$ lies in K. It is clear that, if the point $X' = (x_1, \ldots, x_{n+1})$ belongs to the domain of such a cell of G', then $0 < x_{n+1} < 1$, and for all λ for which $0 < \lambda x_{n+1} < 1$ the point $\lambda X'$ also belongs to the domain of the same cell of G'. Further, if

$$0 < \lambda < 1, \quad 0 < \mu < 1,$$

$$X' = (x_1, \ldots, x_n, 1), \quad Y' = (y_1, \ldots, y_n, 1),$$

then it is clear that the points $\lambda X'$ and $\mu Y'$ belong to the domain of the same cell of G', if and only if the corresponding points X and Y belong to the domain of the same cell of G. Thus K' consists of the closure of the set of all points of the form $\lambda X'$, where $0 \leqslant \lambda \leqslant 1$,

$$X' = (x_1, \ldots, x_n, 1),$$

and X is in K. Now we see that T is the orthogonal projection of K' on the space of points X' with $x_{n+1} = 0$. Since T is closed and K' is the union of a finite number of cells of G', it follows by theorem 9 that T is a complex on some algebraic grating G_1.

LEMMA 10. *Let n be a positive integer, and let R and ϵ be positive numbers. Then there exists a number c, depending only on n, R and ϵ, with the following property. If Λ is any lattice with determinant Δ, with no point X satisfying*

$$0 < |X| < \epsilon,$$

and with a primitive† *lattice point X_1 satisfying*

$$|X_1| < R,$$

then Λ is necessarily generated by X_1 and certain points $X_2, \ldots, X_n$ satisfying

$$|X_r| < c\Delta \quad (r = 1, \ldots, n). \tag{30}$$

Proof. It is clear from the spherical symmetry of the lemma that we may suppose without loss of generality that X_1 has co-ordinates $(0, \ldots, 0, x_n^{(1)})$ where

$$\epsilon \leqslant x_n^{(1)} < R.$$

Let Λ' be the orthogonal projection of Λ on the space $\mathscr{R}'$ with equation $x_n = 0$. Then, as X_1 is a primitive point of Λ, the projection Λ' of Λ is an $(n-1)$-dimensional lattice in $\mathscr{R}'$ with determinant Δ' given by

$$x_n^{(1)} \Delta' = \Delta.$$

Thus

$$\frac{\Delta}{R} < \Delta' \leqslant \frac{\Delta}{\epsilon}.$$

Now, if there were a point $Y' = (y_1, \ldots, y_{n-1}, 0)$ other than O of Λ' with

$$|Y'| < \tfrac{1}{2}\sqrt{(3)}\, \epsilon^2/R, \tag{31}$$

there would be a two-dimensional sublattice of Λ generated by X_1 and a point of the form $Y = (y_1, \ldots, y_n)$ with determinant

$$x_n^{(1)} |Y'| < \tfrac{1}{2}\sqrt{(3)}\, \epsilon^2,$$

and so there would be a point X other than O of this sublattice of Λ satisfying $|X| < \epsilon$. Consequently there is no point Y' other than O of Λ' satisfying (31).

† A lattice point X is said to be primitive if there is no lattice point of the form λX with $0 < \lambda < 1$.

By a well-known result of Minkowski, for each positive integer n, there exists an absolute constant λ_n, such that, if Λ is any lattice in n-dimensional space, then Λ is generated by points $X_1, \ldots, X_n$ satisfying

$$|X_1|.|X_2|\ldots|X_n| < \lambda_n d(\Lambda).$$

Applying this to the lattice Λ', this lattice is generated by points $Y'_1, \ldots, Y'_{n-1}$ satisfying

$$|Y'_1|.|Y'_2|\ldots|Y'_{n-1}| < \lambda_{n-1}\Delta/\epsilon.$$

As $|Y'_r| \geqslant \frac{1}{2}\sqrt{(3)}\,\epsilon^2/R$ for $r = 1, \ldots, n-1$, it follows that

$$|Y'_r| < \lambda_{n-1}\left(\frac{2}{\sqrt{3}}\frac{R}{\epsilon^2}\right)^{n-2}\frac{\Delta}{\epsilon},$$

for $r = 1, \ldots, n-1$. Corresponding to each point Y'_r we can choose a point X_{r+1} of Λ with

$$x_1^{(r+1)} = y_1^{(r)}, \quad \ldots, \quad x_{n-1}^{(r+1)} = y_{n-1}^{(r)}, \quad |x_n^{(r+1)}| \leqslant \tfrac{1}{2}x_n^{(1)}.$$

Then Λ is generated by the points $X_1, X_2, \ldots, X_n$, and we have

$$|X_r| < \lambda_{n-1}\left(\frac{2}{\sqrt{3}}\frac{R}{\epsilon^2}\right)^{n-2}\frac{\Delta}{\epsilon} + R,$$

for $r = 1, \ldots, n$. As there is no point X of Λ other than O in the sphere $|X| < \epsilon$, it follows by a well-known result that $\Delta \geqslant \gamma_n \epsilon^n$, for a suitable positive constant γ_n depending only on n. Hence the points $X_1, \ldots, X_n$ generating Λ satisfy the inequalities (30) with

$$c = \frac{\lambda_{n-1}}{\epsilon}\left(\frac{2}{\sqrt{3}}\frac{R}{\epsilon^2}\right)^{n-2} + \frac{R}{\gamma_n \epsilon^n}.$$

THEOREM 11. *Let S be a proper bounded star set which is also a complex on an algebraic grating G. Let T be the closure of the set of all points of the form λX, where $0 \leqslant \lambda \leqslant 1$ and X is a primitively irreducible point of S. Then T is a proper star set which is also a complex on an algebraic grating.*

Proof. Let G be the grating defined by the polynomial

$$F(X) = F(x_1, \ldots, x_n).$$

We choose R so large that every point X of S satisfies $|X| < R$. By theorem 6, we choose $\epsilon > 0$ so small that every point X with $0 < |X| \leqslant 2\epsilon$ is a primitively irreducible point of S. Let c be the number corresponding to n, R and ϵ, whose existence is established by lemma 10, and write

$$M = c\Delta(S), \quad N = 3c^{n-1}R\{\Delta(S)\}^{n-2}.$$

We use $\mathbf{X}$ to denote the point in n^2-dimensional space with co-ordinates

$$(x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(n)}, \ldots, x_n^{(n)}).$$

We write for convenience $\qquad \mathbf{X} = (X_1, \ldots, X_n).$

We use $\Delta(\mathbf{X})$ to denote the determinant $|x_r^{(s)}|$ of the corresponding matrix. We write

$$\Pi(\mathbf{X}) = \Pi F(u_1 X_1 + \ldots + u_n X_n),$$

the product being taken over all sets of integers $u_1, \ldots, u_n$, not all zero, with

$$|u_r| < N, \quad \text{for } r = 1, \ldots, n.$$

Write

$$H(\mathbf{X}) = \{|X_1|^2 - \epsilon^2\}\{M^2 - |X_1|^2\}\ldots\{M^2 - |X_n|^2\}[\{\Delta(S)\}^2 - 9\{\Delta(\mathbf{X})\}^2][\{\Delta(S)\}^2 - \{\Delta(\mathbf{X})\}^2]\,\Pi(\mathbf{X}).$$

Then $H(\mathbf{X})$ is a polynomial in $x_1^{(1)}, \ldots, x_n^{(n)}$ which is not identically zero. Let G_1 be the grating in n^2-dimensional space defined by this polynomial.

Let K be the complex on the grating G_1 consisting of all cells whose domain contains a point $\mathbf{X} = (X_1, \ldots, X_n)$ with the properties:

(*a*) $|X_1| > \epsilon$;

(*b*) $|X_r| < M \quad (r = 1, \ldots, n)$;

(*c*) $\frac{1}{3}\Delta(S) < |\Delta(\mathbf{X})| < \Delta(S)$;

(*d*) the only points of the lattice Λ generated by $X_1, \ldots, X_n$ which are in S are of the form

$$mX_1 \quad (m = 0, \pm 1, \pm 2, \ldots).$$

Let K' be the projection of K on the space with equations

$$X_2 = X_3 = \ldots = X_n = 0,$$

i.e. let K' be the set of all points X_1 such that there are points $X_2, \ldots, X_n$ for which the point $\mathbf{X} = (X_1, \ldots, X_n)$ is in K. Let C be the closure of K'. It follows by repeated application of theorem 9 that C is a complex on some algebraic grating G_2 say. We prove that C is in fact the closure of the set of all primitively irreducible points X of S with $|X| > \epsilon$.

Let A_1 be any primitively irreducible point of S with $|A_1| > \epsilon$. Then there is a lattice Λ_0 with $d(\Lambda_0) < \Delta(S)$, having A_1 as a lattice point, such that the only points of Λ_0 in S are of the form

$$mA_1 \quad (m = 0, \pm 1, \pm 2, \ldots).$$

By replacing Λ_0 by a suitable sub-lattice of itself, if necessary, we can ensure that $\frac{1}{2}\Delta(S) \leqslant d(\Lambda_0) < \Delta(S)$. Then A_1 is a point of S and is a primitive point of Λ_0. By lemma 10 and our choice of M there are points $A_2, \ldots, A_n$ such that $A_1, \ldots, A_n$ generate Λ_0 and

$$|A_r| < M, \quad \text{for } r = 1, \ldots, n.$$

Consider the point $\mathbf{A} = (A_1, \ldots, A_n)$ in n^2-dimensional space. Note that $d(\Lambda_0) = |\Delta(\mathbf{A})|$, so that

$$\tfrac{1}{3}\Delta(S) < |\Delta(\mathbf{A})| < \Delta(S).$$

Clearly $\mathbf{A}$ is a limit point of points $\mathbf{X}$ which are not on the grating G_1. But provided $\mathbf{X}$ is sufficiently close to $\mathbf{A}$ the conditions (*a*), (*b*), (*c*) and (*d*) above are satisfied. Thus, if $\mathbf{X}$ is sufficiently close to $\mathbf{A}$, and is not on G_1, then $\mathbf{X}$ is in the domain of one of the cells of the complex K, and the projection X_1 of $\mathbf{X}$ is in C. Hence A_1 is a limit point of C. As C is closed it follows that C contains the closure of the set of all primitively irreducible points A_1 of S with $|A_1| > \epsilon$.

Now let X_1 be any point in K'. Then X_1 is the projection of some point $\mathbf{X} = (X_1, \ldots, X_n)$ in K. As $\mathbf{X}$ is in K it is the limit point of points $\mathbf{A}$ in the domain of one of the cells of K. Hence X_1 is a limit point of points A_1, which are projections of points $\mathbf{A}$ in the domain of one of the cells of K. Consider any such point A_1; we prove that A_1 is a primitively irreducible point of S. As the corresponding point $\mathbf{A}$ is in the domain of one of the cells of K, the point $\mathbf{A}$ can be joined by a continuous curve which does not meet G_1 to some point $\mathbf{B}$ such that the point $\mathbf{X} = \mathbf{B}$ has the properties (*a*), (*b*), (*c*) and (*d*) above. Let this curve be given parametrically by

$$\mathbf{X} = \mathbf{X}(t) \quad (0 \leqslant t \leqslant 1).$$

For all t with $0 \leqslant t \leqslant 1$ we have $H(\mathbf{X}(t)) \neq 0$, and so

$$|X_1(t)| \neq \epsilon;$$

$$|X_r(t)| \neq M \quad (r = 1, \ldots, n);$$

$$\tfrac{1}{3}\Delta(S) \neq |\Delta(\mathbf{X}(t))| \neq \Delta(S);$$

$$\Pi(\mathbf{X}(t)) \neq 0.$$

When $t = 1$ the point $\mathbf{X} = \mathbf{X}(1) = \mathbf{B}$ satisfies the conditions (a), (b) and (c), thus by continuity considerations when $t = 0$ the point $\mathbf{X} = \mathbf{X}(0) = \mathbf{A}$ satisfies the conditions (a), (b) and (c). Let Λ_0 be the lattice generated by $A_1, \ldots, A_n$. Then $\frac{1}{3}\Delta(S) < d(\Lambda_0) = |\Delta(\mathbf{A})| < \Delta(S)$. We also have

$$|A_r| < M, \quad \text{for } r = 1, \ldots, n.$$

Let A be any point of Λ_0 which is in S. Then $|A| < R$ and, by lemma 1 and our choice of M and N,

$$A = u_1 A_1 + \ldots + u_n A_n$$

for some integers $u_1, \ldots, u_n$ satisfying

$$|u_r| < N \quad (r = 1, \ldots, n).$$

Thus, as $\Pi(\mathbf{A}) \neq 0$, we have

$$F(A) = F(u_1 A_1 + \ldots + u_n A_n) \neq 0,$$

and A is in the domain of one of the cells of the complex S. But we also have

$$F(u_1 X_1(t) + \ldots + u_n X_n(t)) \neq 0,$$

since $\Pi(\mathbf{X}(t)) \neq 0$, for $0 \leqslant t \leqslant 1$. So the curve given by

$$X(t) = u_1 X_1(t) + \ldots + u_n X_n(t) \quad (0 \leqslant t \leqslant 1),$$

is a continuous curve, which does not meet G, leading from the point A to the point

$$B = u_1 B_1 + \ldots + u_n B_n.$$

Consequently this point B is a point of the domain of one of the cells of S. As the point $\mathbf{X} = \mathbf{B}$ satisfies the condition (d) above, we must have

$$B = u_1 B_1,$$

and

$$u_2 = u_3 = \ldots = u_n = 0.$$

Thus $A = u_1 A_1$ and A is of the form

$$mA_1 \quad (m = 0, \pm 1, \pm 2, \ldots).$$

This proves that A_1 is a primitively irreducible point of S with $|A_1| > \epsilon$. It follows that every point of K', and therefore every point of C, is a limit point of the set of primitively irreducible points X of S satisfying $|X| > \epsilon$. Hence by the last paragraphs the algebraic complex C on G_2 coincides with the closure of the set of primitively irreducible points X of S satisfying $|X| > \epsilon$.

As every point X with $\epsilon < |X| \leqslant 2\epsilon$ is a primitively irreducible point of S, it is clear that T is the closure of the set of all points of the form λX, where $0 \leqslant \lambda \leqslant 1$ and X is in C. It follows by theorem 10 that T is a complex on some algebraic grating G_3. It is clear T is a proper star set. This proves the theorem.

We now prove a lemma analogous to lemma 8.

LEMMA 11. *Let S be a proper bounded star set, which is a complex on an algebraic grating G. Suppose that all points X with $0<|X|\leqslant\epsilon$ are primitively irreducible points of S, and that $|X|<R$ for all points X of S. Suppose that for some point X_0, with $0<|X_0|\leqslant R$, all the outer boundary points X of S with*

$$\left.\begin{aligned}|X|>|X_0|,\\ \left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}\quad(\textit{for some sign}),\end{aligned}\right\}\tag{32}$$

are irreducible. Then there is a proper star set T which is a complex on an algebraic grating, which is contained in S and has $\Delta(T)=\Delta(S)$, and which is such that all the outer boundary points X of T with

$$\left.\begin{aligned}|X|>|X_0|-\tfrac{1}{4}\epsilon,\\ \left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|<\frac{\epsilon}{4R}\quad(\textit{for some sign}),\end{aligned}\right\}\tag{33}$$

are irreducible.

Proof. We take T_0 to be the union of the set T_1 of all points X of S such that either

$$|X|\leqslant|X_0|-\tfrac{1}{4}\epsilon$$

or

$$\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|\geqslant\frac{\epsilon}{4R}\quad\text{(for both signs)},$$

and of the closure T_2 of the set of all points of the form λX where $0\leqslant\lambda\leqslant 1$ and X is a primitively irreducible point of S. Clearly T_2 is also the set of all points of the form λX, where $0\leqslant\lambda\leqslant 1$ and X is an irreducible point of S. It follows by the proof of lemma 8 that T_0 is a star set contained in S with $\Delta(T_0)=\Delta(S)$ and such that all the outer boundary points X of T_0 satisfying (33) are irreducible points of T_0. We take T to be the closure of the set of inner points of T_0. Then T is clearly a proper star set contained in S, and by theorem 8 we have $\Delta(T)=\Delta(T_0)=\Delta(S)$. Further as T is contained in T_0, all the outer boundary points X of T satisfying (33) are irreducible. Now we have only to prove that T is a complex on some algebraic grating.

Let the grating G, on which S is a complex, be defined by the polynomial

$$F(X)=F(x_1,\ldots,x_n).$$

By theorem 11 the set T_2 is a complex on an algebraic grating G_2 defined by some polynomial

$$F_2(X)=F_2(x_1,\ldots,x_n),$$

say. Write

$$F_1(X)=[|X|^2-\{|X_0|-\tfrac{1}{4}\epsilon\}^2]\left[\left\{1-\frac{1}{2}\left(\frac{\epsilon}{4R}\right)^2\right\}^2|X|^2|X_0|^2-\left\{\sum_{r=1}^{n}x_r x_r^{(0)}\right\}^2\right];$$

so that $F_1(X)=0$ if either

$$|X|=|X_0|-\tfrac{1}{4}\epsilon$$

or

$$\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|=\frac{\epsilon}{4R}\quad\text{(for some sign)}.$$

Take G_3 to be the grating defined by the polynomial

$$F_3(X)=F(X)\,F_1(X)\,F_2(X).$$

Now it is clear that T is the union of the set T_2 and the closure T_1' of the set of all inner points X of S satisfying either

$$|X|<|X_0|-\tfrac{1}{4}\epsilon\tag{34}$$

or

$$\left|\frac{X}{|X|}\pm\frac{X_0}{|X_0|}\right|>\frac{\epsilon}{4R}\quad\text{(for both signs)}.\tag{35}$$

But as T_2 is a complex on G_2 and as G_3 includes G_2, it follows that T_2 is a complex on G_3. Also as S is a complex on G and G_3 includes G and the points X with $F_1(X) = 0$, it follows that T_1' is a complex on G_3, the complex on G_3 consisting of the cells of G_3 whose domains contain an inner point X of S satisfying either (34) or (35). Hence T, being the union of two complexes on G_3, is itself a complex on G_3. This proves the lemma.

We are now in a position to prove the main result of this section, namely theorem 12 stated in § 1.

Proof of theorem 12. We recall that lemma 9 is proved by a finite number of applications of lemma 8 and that theorem 7 is proved by a finite number of applications of lemma 9. By a precisely similar use of a finite number of applications of lemma 11 we may prove a lemma analogous to lemma 9, which may be used a finite number of times, as in the proof of theorem 7, in order to prove theorem 12. This procedure suffices to prove the theorem.

We conclude by giving the proof of theorem 13, stated in § 1.

Proof of theorem 13. The bodies in the table have the critical determinants shown by the work of Minkowski (1904) for S_1, Davenport (1941, 1939) for S_2 and S_3, Markoff (1903) for S_4, and Oppenheim (see Dickson 1930) for S_5 and S_6.

The body S_1 is clearly bounded and is a complex on the grating with equation

$$\Pi(1 \pm x_1 \pm x_2 \pm x_3) = 0,$$

the product being taken over all possible combinations of the signs. The result for this body follows directly from theorem 11.

For the unbounded bodies $S_2, \ldots, S_6$ we need results of Mahler. Let $S_r^{(t)}$ be the set of all points X of S_r with

$$|X| \leqslant t,$$

for $r = 2, \ldots, 6$ and any positive t. Then $S_r^{(t)}$ is a bounded star body and Mahler (1946*b*, §§ 14, 15 and 16)† has proved that

$$\Delta(S_r^{(t)}) = \Delta(S_r),$$

for $r = 2, \ldots, 6$, provided that t is sufficiently large. Now for $r = 2, \ldots, 6$ the body $S_r^{(t)}$ is both a bounded star body and a complex on an algebraic grating, for example $S_2^{(t)}$ is one of the cells of the grating with equation

$$(t^2 - x_1^2 - x_2^2 - x_3^2)(1 - x_1^2 x_2^2 x_3^2) = 0.$$

It follows by theorem 12 that provided t is sufficiently large there is for $r = 2, \ldots, 6$ a proper bounded star set T with

$$\Delta(T_r) = \Delta(S_r^{(t)}) = \Delta(S_r),$$

which is a complex on an algebraic grating, and which is irreducible among the star sets. This completes the proof.

Appendix 1. Comparison of two definitions for the critical determinant

In this appendix we use $\Delta_M(S)$ to denote the value assigned to the critical determinant of S by Mahler's definition, and we use $\Delta(S)$ to denote the value assigned to the critical determinant of S by our definition. It is clear from the definitions that

$$\Delta_M(S) \leqslant \Delta(S)$$

† But for S_3 see also Davenport & Rogers (1950).

for all sets S. We give three examples of sets S for which

$$\Delta_M(S) < \Delta(S).$$

Example 1. Consider the 'square frame' S of points (x, y) in two-dimensional space satisfying†

$$1 \leqslant \max(|x|, |y|) \leqslant 2.$$

If X is any point other than O of a lattice Λ satisfying

$$\max(|x|, |y|) \leqslant 2,$$

then it is clear that there is a point of Λ of the form mX, where $m = \pm 1, \pm 2, \ldots$, in S. Thus, using our definition, the determinant of S is the same as that of the set

$$\max(|x|, |y|) \leqslant 2,$$

and so

$$\Delta(S) = 4.$$

But the lattice of points with integral co-ordinates has no point in the interior of S, and so using Mahler's definition

$$\Delta_M(S) \leqslant 1$$

(and in fact it is easy to see that $\Delta_M(S) = 1$).

Example 2. Take S to be the set formed by adding to the set of all points $X = (x, y)$ satisfying

$$\min\{|x^2 - xy - y^2|, |y^2 - yx - x^2|\} \leqslant 1,$$

the set of all points (x, y) satisfying

$$y = 0, \quad 1 < |x| \leqslant \sqrt[4]{(1\cdot 6)}.$$

Then it follows from a result of mine‡ that

$$\Delta(S) = \sqrt{(1\cdot 6)}.$$

On the other hand, the lattice of points with integral co-ordinates has no point other than O in the interior of S, so that

$$\Delta_M(S) \leqslant 1$$

(and again $\Delta_M(S) = 1$ in fact). Note that this set S is a star set.

Example 3. Take S to be the closed bounded star set bounded by the line segments joining the points

$$\begin{array}{llllll}
(1, 0), & (1\tfrac{1}{2}, 0), & (2\tfrac{1}{2}, 1), & (2, 1), & (2, 2), & (1, 1), \\
(0, 1), & (0, 1\tfrac{1}{2}), & (-1, 2\tfrac{1}{2}), & (-1, 2), & (-2, 2), & (-1, 1), \\
(-1, 0), & (-1\tfrac{1}{2}, 0), & (-2\tfrac{1}{2}, -1), & (-2, -1), & (-2, -2), & (-1, -1), \\
(0, -1), & (0, -1\tfrac{1}{2}), & (1, -2\tfrac{1}{2}), & (1, -2), & (2, -2), & (1, -1), \\
(1, 0), & & & & &
\end{array}$$

in this order (see figure 1). Note that this set S is a proper bounded star set.

It is clear from the figure that there is no point other than O with integral co-ordinates in the interior of S. On the other hand S contains the square given by

$$|x| \leqslant 1, \quad |y| \leqslant 1.$$

Consequently

$$\Delta_M(S) = 1.$$

† Such frames have been discussed in detail by Ollerenshaw (1944, 1945*b*).
‡ Rogers (1947*b*, theorem 1).

We prove that $\Delta(S) = 1\frac{1}{2}$. Consider any lattice with $d(\Lambda) < 1\frac{1}{2}$ and with no point other than O in the interior of S. As $d(\Lambda) < 1\frac{1}{2}$, there is a point other than O of Λ in the interior of the parallelogram with vertices at the points $\pm(1, \frac{1}{2})$, $\pm(-1, 2\frac{1}{2})$. Since there is no point other than O of Λ in the interior of S, it follows that there is a point X_1 of Λ with co-ordinates (x_1, y_1), where

$$x_1 \geq 0, \quad y_1 \geq 1, \quad x_1 + y_1 < 1\tfrac{1}{2}.$$

Similarly, there is a point X_2 of Λ with co-ordinates (x_2, y_2), where

$$x_2 \geq 1, \quad y_2 \leq 0, \quad x_2 - y_2 < 1\tfrac{1}{2}.$$

As the point $X_1 + X_2$ is a point of Λ and

$$1 \leq x_1 + x_2 < 2, \quad \tfrac{1}{2} < y_1 + y_2 < 1\tfrac{1}{2},$$

this point is not an inner point of S, and we must have

$$y_1 + y_2 \geq x_1 + x_2.$$

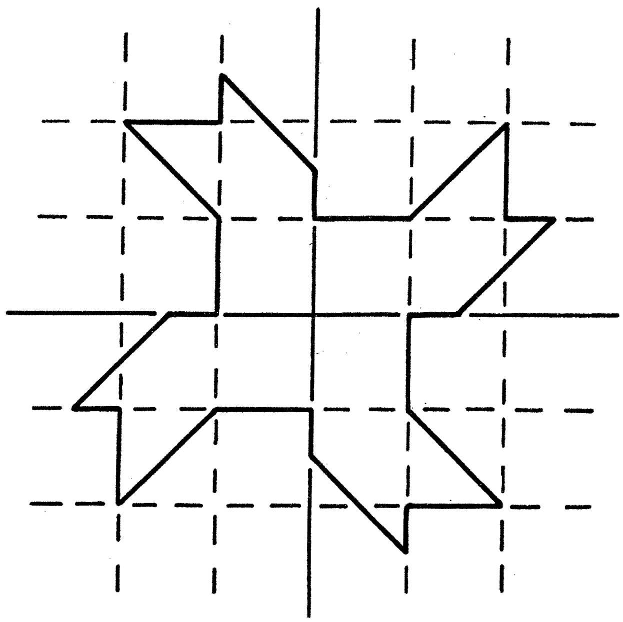

FIGURE 1

Similarly the point $X_2 - X_1$ is not an inner point of S and

$$x_2 - x_1 \geq -(y_2 - y_1).$$

But these last inequalities imply that

$$2x_1 - 2y_2 = (x_1 + x_2) - (y_1 + y_2) - (y_2 - y_1) - (x_2 - x_1) \leq 0.$$

Consequently $x_1 = y_2 = 0$ and using these results

$$y_1 \geq x_2, \quad x_2 \geq y_1.$$

Hence $x_2 = y_1$ and the points $(0, y_1)$, $(y_1, 0)$ are points of Λ. The determinant of the lattice generated by these points is

$$y_1^2 < (1\tfrac{1}{2})^2 \leq 2\tfrac{1}{4} d(\Lambda).$$

But the mid-point $(\frac{1}{2}y_1, \frac{1}{2}y_1)$ of $(0, y_1)$ and $(y_1, 0)$ is in the interior of S. Thus Λ is generated by the points $(0, y_1)$, $(y_1, 0)$. Now we have shown that every lattice Λ with $d(\Lambda) < 1\frac{1}{2}$, either has a point other than O in the interior of S, or is a multiple of the lattice of points with integral co-ordinates, in which case Λ has points other than O on the boundary of S. This proves that $\Delta(S) \geqslant 1\frac{1}{2}$. But it is easy to verify that, if ϵ is sufficiently small and positive then the lattice generated by the points

$$(1+2\epsilon, 1+2\epsilon+\epsilon^2), \quad (0, 1\tfrac{1}{2}+3\epsilon)$$

has determinant $1\frac{1}{2}(1+2\epsilon)^2$ and has no point other than O in S. Hence $\Delta(S) \leqslant 1\frac{1}{2}$. This completes the proof that $\Delta(S) = 1\frac{1}{2}$.

Appendix 2. A star domain containing no irreducible star domain with the same determinant

In this appendix we given an example of a bounded star domain† S with the property that, if S' is any star domain contained in S with $\Delta(S') = \Delta(S)$, then there is a star domain S'' properly contained in S' with $\Delta(S'') = \Delta(S') = \Delta(S)$. Before we construct this example it is convenient to find the critical determinant of a certain set S_0.

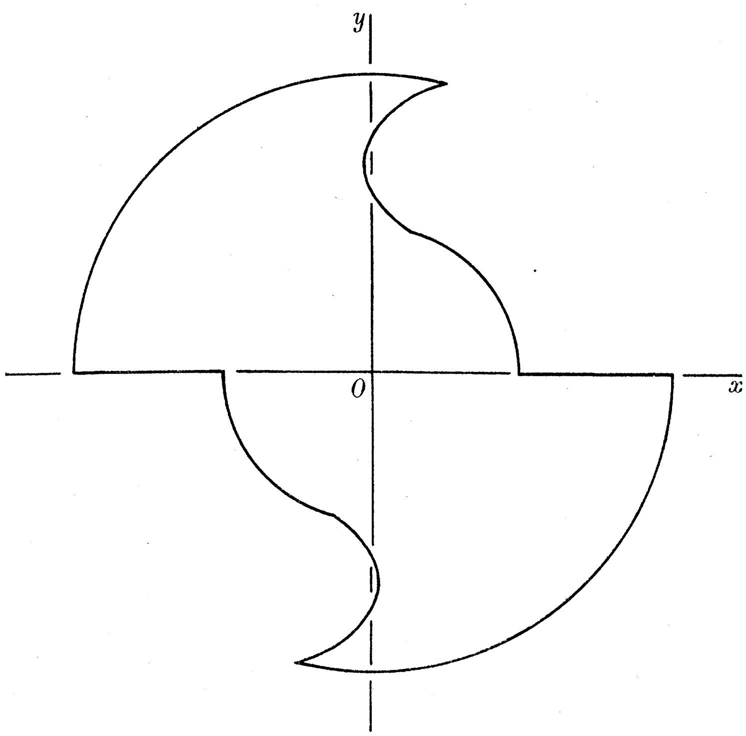

Figure 2

Let S_0 be the closed set containing the origin O and bounded by the curves given parametrically by:

$$x = 2-t, \quad y = 0, \quad (0 \leqslant t \leqslant 1);$$
$$x = \cos\theta, \quad y = \sin\theta, \quad (0 \leqslant \theta \leqslant \cos^{-1}\tfrac{1}{4});$$
$$x = \tfrac{1}{4}\sqrt{(15)}\operatorname{cosec}\phi - 2\cos\phi, \quad y = 2\sin\phi, \quad (\cos^{-1}\tfrac{7}{8} \leqslant \phi \leqslant \cos^{-1}\tfrac{1}{4});$$
$$x = 2\cos\theta, \quad y = 2\sin\theta, \quad (\cos^{-1}\tfrac{1}{4} \leqslant \theta \leqslant \pi);$$

† A star domain is a two-dimensional star body.

and the corresponding curves obtained by rotating these curves about O through two right angles (see figure 2). The third of these curves is not very simple, and needs some investigation. It is easy to verify that for this curve

$$x^2+y^2 = \frac{15}{16}\left\{1+\left(\frac{8}{\surd(15)}-\cot\phi\right)^2\right\},$$

$$\frac{x}{y} = \frac{\surd(15)}{8}\left\{\left(\cot\phi-\frac{4}{\surd(15)}\right)^2-\frac{1}{15}\right\}.$$

Thus, as ϕ increases from $\cos^{-1}\frac{7}{8}$ to $\cos^{-1}(4/\surd(31))$, $\cot\phi$ decreases from $7/\surd(15)$ to $4/\surd(15)$, x^2+y^2 increases from 1 to $1\frac{15}{16}$, and x/y decreases from $1/\surd(15)$ to $-1/8\surd(15)$; and as ϕ increases from $\cos^{-1}(4/\surd(31))$ to $\cos^{-1}\frac{1}{4}$, $\cot\phi$ decreases from $4/\surd(15)$ to $1/\surd(15)$, x^2+y^2 increases from $1\frac{15}{16}$ to 4, and x/y increases from $-1/8\surd(15)$ to $1/\surd(15)$.

We prove that $\Delta(S_0) = \frac{1}{2}\surd(15)$. In the first place the lattice Λ_0 generated by the points $(0, 2)$ and $(\frac{1}{4}\surd(15), \frac{1}{4})$ has no point other than O in the interior of S_0, and for all $\epsilon>0$ the lattice $(1+\epsilon)\Lambda_0$ has no point other than O in S_0. Hence $\Delta(S_0)\leqslant\frac{1}{2}\surd(15)$. Now suppose that Λ is any lattice with no point other than O in the interior of S_0 and with $d(\Lambda)\leqslant\frac{1}{2}\surd(15)$. Then there is a point other than O of Λ in the circle

$$x^2+y^2<4,$$

since the critical determinant of this circle is $2\surd3>\frac{1}{2}\surd(15)$. So by rotating Λ clockwise through a suitable angle we obtain a lattice Λ_1, which has no point other than O in the interior of S_0, which has a lattice point with co-ordinates $(\xi, 0)$ satisfying $1\leqslant\xi<2$, and which has determinant

$$d(\Lambda_1) = d(\Lambda)\leqslant\tfrac{1}{2}\surd(15).$$

Write

$$\alpha = \sin^{-1}\frac{\surd(15)}{4\xi} = \cos^{-1}\frac{\surd(16\xi^2-15)}{4\xi}.$$

Then

$$\cos^{-1}\tfrac{7}{8}<\alpha\leqslant\cos^{-1}\tfrac{1}{4},$$

since $1\leqslant\xi<2$. Write

$$\beta = \sin^{-1}\frac{d(\Lambda_1)}{2\xi}.$$

Then there are an infinite number of points of Λ_1 at a distance ξ apart on the line l with equation

$$y = \frac{d(\Lambda_1)}{\xi} = 2\sin\beta.$$

As $d(\Lambda_1)\leqslant\frac{1}{2}\surd(15)$, we have

$$\beta\leqslant\sin^{-1}\frac{\surd(15)}{4\xi} = \alpha.$$

If β were less than or equal to $\cos^{-1}\frac{7}{8}$, the line l would meet the interior of S_0 in an interval of length

$$\begin{aligned} 2\cos\beta+\surd(1-4\sin^2\beta) &= 2\cos\beta+\surd(4\cos^2\beta-3) \\ &\geqslant 2(\tfrac{7}{8})+\surd\{4(\tfrac{7}{8})^2-3\} \\ &= 2>\xi. \end{aligned}$$

This is impossible as every interval of length greater than ξ of l contains a point of Λ_1. Hence

$$\cos^{-1}\tfrac{7}{8}<\beta\leqslant\alpha\leqslant\cos^{-1}\tfrac{1}{4},$$

and the line l meets the interior of S_0 in an interval of length

$$\tfrac{1}{4}\sqrt{(15)}\operatorname{cosec}\beta \geqslant \tfrac{1}{4}\sqrt{(15)}\operatorname{cosec}\alpha = \xi.$$

This length is greater than ξ unless $\alpha = \beta$. Thus we must have $\alpha = \beta$; and Λ_1 is generated by the points $(\frac{1}{4}\sqrt{(15)}\operatorname{cosec}\alpha, 0)$ and $(-2\cos\alpha, 2\sin\alpha)$, where $\cos^{-1}\frac{7}{8} < \alpha \leqslant \cos^{-1}\frac{1}{4}$. Now we have

$$d(\Lambda) = d(\Lambda_1) = \tfrac{1}{2}\sqrt{(15)}.$$

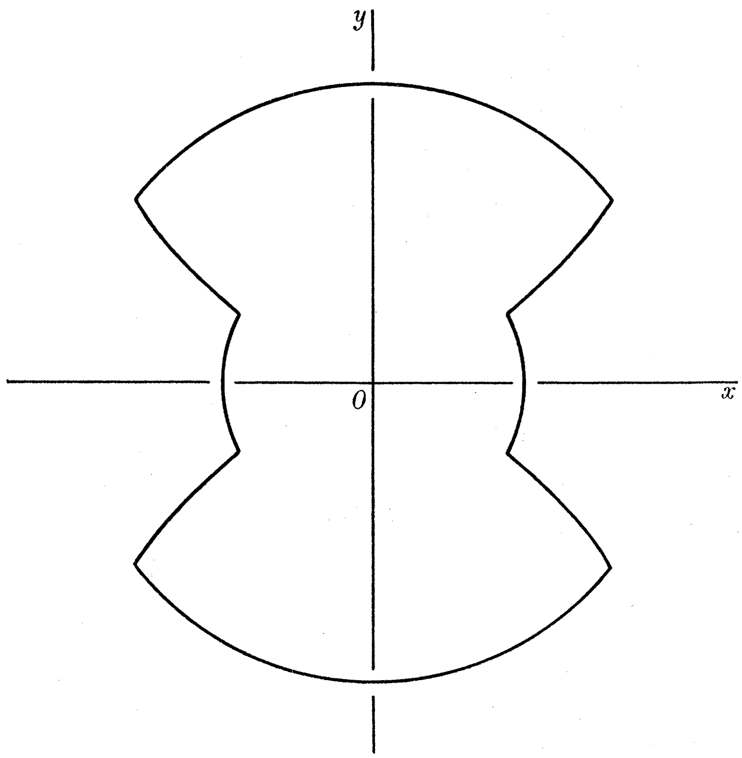

FIGURE 3

As Λ_1 was obtained from Λ by a rotation about O, it is clear that either

$$\cos^{-1}\tfrac{7}{8} < \alpha < \cos^{-1}\tfrac{1}{4}$$

and $\Lambda = \Lambda_1$ is the lattice generated by the points $(\frac{1}{4}\sqrt{(15)}\operatorname{cosec}\alpha, 0)$, $(-2\cos\alpha, 2\sin\alpha)$ or $\alpha = \cos^{-1}\frac{1}{4}$ and Λ is the lattice generated by the points

$$(\cos\theta, \sin\theta), \quad (-\tfrac{1}{2}\cos\theta - \tfrac{1}{2}\sqrt{(15)}\sin\theta,\ -\tfrac{1}{2}\sin\theta + \tfrac{1}{2}\sqrt{(15)}\cos\theta)$$

for some θ with $0 \leqslant \theta \leqslant \cos^{-1}\frac{1}{4}$. It is easy to verify that the only lattice Λ, which is of one of these forms, and which has a lattice point on the circle

$$x^2 + y^2 = 1\tfrac{15}{16}, \tag{36}$$

is the lattice generated by the points

$$(\tfrac{1}{4}\sqrt{(31)}, 0), \quad \left(\frac{-1}{4\sqrt{(31)}}, 2\sqrt{\left(\frac{15}{31}\right)}\right). \tag{37}$$

We have now proved that $\Delta(S_0) = \frac{1}{2}\sqrt{(15)}$, and that, *if Λ is any lattice with $d(\Lambda) \leqslant \frac{1}{2}\sqrt{(15)}$ having a lattice point on the circle* (36) *and having no point other than O in the interior of S_0, then Λ is the lattice generated by the points* (37).

Take S to be the star domain bounded by the curves given parametrically by:

$$x = \cos\theta, \qquad y = \sin\theta \qquad (0 \leqslant \theta \leqslant \chi - \epsilon);$$
$$x = (1+t)\cos(\chi-\epsilon+t\epsilon), \quad y = (1+t)\sin(\chi-\epsilon+t\epsilon) \quad (0 \leqslant t \leqslant 1);$$
$$x = 2\cos\theta, \qquad y = 2\sin\theta \qquad (\chi \leqslant \theta \leqslant \tfrac{1}{2}\pi)$$

and the reflexions of these curves in the x-axis, the y-axis and both axes, where

$$\chi = \tfrac{1}{2}\cos^{-1}\tfrac{1}{4}$$

and ϵ is a suitable small positive angle (see figure 3). Let S_θ be the set obtained by rotating S_0 clockwise about O through an angle θ. Then it is clear that S contains the set S_χ. Hence we have

$$\Delta(S) \geqslant \Delta(S_\chi) = \Delta(S_0) = \tfrac{1}{2}\sqrt{(15)}.$$

But the lattice generated by the points

$$(1, 0) \quad \text{and} \quad (\tfrac{1}{2}, \tfrac{1}{2}\sqrt{(15)})$$

has determinant $\frac{1}{2}\sqrt{(15)}$ and has no point other than O in the interior of S. Consequently $\Delta(S) = \frac{1}{2}\sqrt{(15)}$.

Let S' be any star domain contained in S with $\Delta(S') = \Delta(S)$. To prove our result, we suppose that there is no star domain S'' properly contained in S' with $\Delta(S'') = \Delta(S') = \Delta(S)$, and we obtain a contradiction. Then S' is irreducible among the star domains. It is easy to verify that the points given parametrically by

$$x = \cos\theta, \quad y = \sin\theta \quad (0 \leqslant \theta < 2\pi)$$

and by

$$x = 2\cos\theta, \quad y = 2\sin\theta \quad (\chi+\epsilon \leqslant \theta \leqslant \pi-\chi-\epsilon)$$

are irreducible points of S. Consequently S' contains all these points, and in particular the points $(\cos(\chi-\epsilon), \sin(\chi-\epsilon))$ and $(2\cos(\chi+\epsilon), 2\sin(\chi+\epsilon))$ are on the boundary of S'. These points on the boundary of S' have polar co-ordinates $(1, \chi-\epsilon)$ and $(2, \chi+\epsilon)$. As S' is a star domain it follows that there is a point $X_1 = (x_1, y_1)$ with polar co-ordinates (r_1, θ_1), satisfying

$$r_1 = \sqrt{1\tfrac{15}{16}}, \quad \chi-\epsilon < \theta_1 < \chi+\epsilon,$$

which is on the boundary of S'. Since S' is an irreducible star domain, it follows by a result of Mahler (1946*b*, theorem C) that there is a critical lattice Λ_1 of S' with X_1 as a lattice point.

Now the point $(\frac{1}{4}\sqrt{(31)}\cos(\chi+\epsilon), -\frac{1}{4}\sqrt{(31)}\sin(\chi+\epsilon))$ is an inner point of S', as the point $(2\cos(\chi+\epsilon), -2\sin(\chi+\epsilon))$ is a boundary point of S'. Thus Λ_1 is a lattice with $d(\Lambda_1) = \frac{1}{2}\sqrt{(15)}$ with a lattice point X_1 on the circle (36), but which does not have the point

$$(\tfrac{1}{4}\sqrt{(31)}\cos(\chi+\epsilon), -\tfrac{1}{4}\sqrt{(31)}\sin(\chi+\epsilon))$$

as a lattice point. It follows from the result italicized above that there is a point other than O of Λ_1 in the interior of $S_{\chi+\epsilon}$. As there is no point other than O of Λ_1 in the interior of the star domain S', it follows that there is a point X_2 with polar co-ordinates (r_2, θ_2), satisfying

$$1 \leqslant r_2 \leqslant 2, \quad \chi-\epsilon \leqslant \theta_2 \leqslant \chi+\epsilon,$$

which is an inner point of $S_{\chi+\epsilon}$. Provided ϵ is sufficiently small, the point X_1 is not in $S_{\chi+\epsilon}$ and so the points X_1 and X_2 are distinct. Thus, provided ϵ is sufficiently small, the point $X_2 - X_1$ is a point other than O of Λ_1, which is in the interior of the circle

$$x^2 + y^2 \leqslant 1$$

and which is therefore an inner point of S'. Hence Λ_1 is not a critical lattice of S'. This contradiction proves the required result.

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References

Cassels, J. W. S. 1947 *Quart. J. Math.* **18**, 236–243.
Davenport, H. 1939 *Proc. Lond. Math. Soc.* (2), **45**, 98–125.
Davenport, H. 1941 *J. Lond. Math. Soc.* **16**, 98–101.
Davenport, H. & Rogers, C. A. 1950 *Phil. Trans.* A, **242**, 311–344.
Dickson, L. E. 1930 *Studies in the theory of numbers*. Chicago: The University of Chicago Press.
Mahler, K. 1944 *Proc. Camb. Phil. Soc.* **40**, 107–116.
Mahler, K. 1946*a* *Proc. Roy. Soc.* A, **187**, 151–187.
Mahler, K. 1946*b* *Proc. Acad. Sci. Amst.* **49**, 331–343, 444–454, 524–532, 622–631.
Mahler, K. 1947 *Proc. Acad. Sci. Amst.* **50**, 98–107.
Mahler, K. 1949 *Canad. J. Math.* **1**, 78–87.
Markoff, A. 1903 *Math. Ann.* **56**, 233–251.
Minkowski, H. 1904 *Nachr. Ges. Wiss. Göttingen*, pp. 311–335.
Newman, M. H. A. 1939 *Elements of the topology of plane sets of points*. Cambridge University Press.
Mullineux, N. 1951 *Proc. Lond. Math. Soc.* (2), **54**, 1–41.
Ollerenshaw, K. 1944 *J. Lond. Math. Soc.* **19**, 178–184.
Ollerenshaw, K. 1945*a* *Quart. J. Math.* **17**, 223–239.
Ollerenshaw, K. 1945*b* *J. Lond. Math. Soc.* **20**, 22–26.
Ostrowski, A. 1920 *Nachr. Ges. Wiss. Göttingen, Materialien für eine Biographie von Gauss*, **8**, 50–58.
Rogers, C. A. 1947*a* *Proc. Acad. Sci. Amst.* **50**, 868–872.
Rogers, C. A. 1947*b* *Proc. Roy. Soc.* A, **191**, 503–517.
van der Waerden, B. L. 1930 *Math. Ann.* **102**, 335–362, (360–361).
van der Waerden, B. L. 1940 *Moderne Algebra*. Berlin: Springer.

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